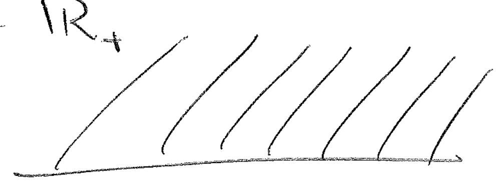
Tuomas Hytönen 09/06/2015.

Basic problem/setup.

Given f on $\partial\Omega = \mathbb{R}^n$, find u in $\Omega = \mathbb{R}^{n+1}_+$

s.t. "$u|_{\partial\Omega}$" $= f$



Ex. $u(x,t) = \int_{\mathbb{R}^n} P_t(x-y) f(y)\, dy$. $P_t(x) = c_n \frac{t}{(|t|^2+|x|^2)^{\frac{n+1}{2}}}$.

if $f \in L^p(\mathbb{R})$ $\quad 1<p<\infty$.

Then $u(x,t) \longrightarrow \begin{cases} f(x). & t \to 0. \quad \text{a.e. } x. \\ 0. & t \to \infty. \quad " . \end{cases}$

$$f(x) = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} u(x,\varepsilon) - u(x,R) = -\int_0^\infty \partial_t u(x,t)\, dt.$$

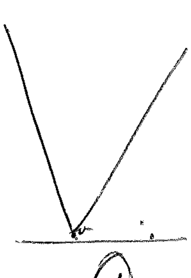
(*) Utopia: $\int_0^\infty |\partial_t u(x,t)|\, dt < \infty$.

$$\Big\| \int_0^\infty t\,|\partial_t u(\cdot,t)| \frac{dt}{t} \Big\|_{L^p} \eqsim \|f\|_{L^p}. \quad \text{FALSE!}$$

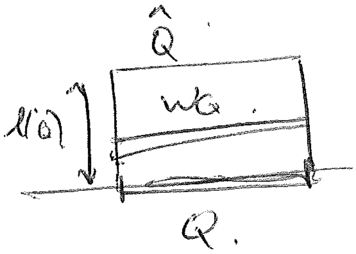
Reality: (Littlewood-Paley).

$$\Big\| \Big(\int_0^\infty |t\, \partial_t u(\cdot,t)|^2 \frac{dt}{t} \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n)} \eqsim \|f\|_{L^p(\mathbb{R}^n)}.$$

$$\eqsim \Big\| \Big(\iint_{|y-x|<t} |t\,\partial_t u(y,t)|^2 \frac{dt\, dy}{t^{n+1}} \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n, dx)}.$$



Ex. "Dyadic Poisson ext"



$\mathcal{D} = \{\text{dyadic cubes } Q\}$, $\quad \hat{Q} := Q \times (0, \ell(Q)]$

$$W_Q = Q \times \left(\tfrac{1}{2}\ell(Q), \ell(Q)\right] \qquad \text{"Whitney box".}$$

$$\hat{R} = \bigcup_{\substack{Q \subset R \\ Q \in \mathcal{D}}} W_Q .$$

$u|_{W_Q} := \langle f \rangle_Q$ ← average over Q.

$$f(x) = \sum_{Q \ni x} \left[\langle f \rangle_Q - \langle f \rangle_{Q^1} \right].$$

← parent cube of Q.

(*) Dyadic version of

$$\sum_{Q \in \mathcal{D}} \left| \langle f \rangle_Q - \langle f \rangle_{Q^1} \right| \chi_Q(x) < \infty .$$

$$\Big\| \sum_{Q \in \mathcal{D}} \left| \langle f \rangle_Q - \langle f \rangle_{Q^1} \right| \chi_Q(x) \Big\|_{L^p} \lesssim \|f\|_{L^p}.$$

$$\Big\| \Big(\sum_{Q \in \mathcal{D}} \left| \langle f \rangle_Q - \langle f \rangle_{Q^1} \right|^2 \chi_Q \Big)^{\frac{1}{2}} \Big\|_{L^p} \simeq \|f\|_{L^p}.$$

Little ℓ^2 norm is smaller than little ℓ^1 norm.
This is ~~not~~ not so apparent in (*) cts version.

End point, $p = \infty$, $f \in BMO$,

$$\sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |f - \langle f \rangle_Q| = \|f\|_{BMO} < \infty .$$

u = Poisson ext.

$$\Big\| \sup_{Q\ni x} \Big(\frac{1}{|Q|} \iint_{\hat Q} |t \nabla u(y,t)|^2 \frac{dy\,dt}{t} \Big)^{\frac12} \Big\|_{L^\infty(\mathbb{R}^n,\, dx)} \simeq \|f\|_{BMO}.$$

Th^m (Varopoulos 77–78).

$\exists$ an extension u s.t.

$$\Big\| \sup_{Q\ni x} \frac{1}{|Q|} \iint_{\hat Q} t|\nabla u| \frac{dt}{\cancel{t}}\, dy \Big\|_{L^\infty} \lesssim \|f\|_{BMO}$$

$$\Big(\sup_{Q} \frac{1}{|Q|} \iint_{\hat Q} |\nabla u|\, dy\,dt \Big\} \lesssim \|f\|_{BMO} \Big)$$

← here [illegible] generalisation of L^p problems, original Varopoulos formulation?

Th^m 1 $\forall f \in L^p$, $1<p<\infty$, $\exists$ ext. u s.t.

$$\|C(|\nabla u|)\|_{L^p} \lesssim \|f\|_{L^p}$$

$$C_\mu(x) := \sup_{Q\ni x} \frac{1}{|Q|} \iint_{\hat Q} d|\mu|(y,t)$$

← defined for measures, and in particular $|\nabla u|$.

Th^m 2 (ε-approx).

$\forall f \in L^p$, $\forall \varepsilon>0$, if u = dyadic Poisson ext of f, then $\exists v$ in $\mathbb{R}^{n+1}_+$:

$$N_d(v-u)(x) := \sup_{|y-x|<t} \|v|_{\omega_Q} - u|_{\omega_Q}\| \quad \text{satisfies}$$

$$\begin{cases} \|N_d(v-u)\|_{L^p} \le \varepsilon \|f\|_{L^p} \\ \|C_d(|\nabla v|)\|_{L^p} \lesssim C_\varepsilon \|f\|_{L^p} \end{cases} \qquad \Big(C_\varepsilon \le c \cdot \frac{1}{\varepsilon} \Big)$$