T. Lamm — Rigidity & non-rigidity for conformal immersions. 05/03/2015.

Thm (De Lellis – Müller '05 ($n=3$), L. Schätzle '14 ($n\geq 3$)).

$\Sigma^2 \subset \mathbb{R}^n$ smooth, closed. $|\Sigma| = 4\pi$.

$\int_\Sigma |\mathring{A}|^2 < 4\pi \Rightarrow \exists$ conformal parametrization $f: S^2 \to \Sigma$.

$\mathring{A}_{ij} = A_{ij} - \frac{1}{2} H g_{ij}$. $\qquad C_\Sigma \in \mathbb{R}^n$.

$$\|f - (C_\Sigma + id_{S^2})\|_{W^{2,2}} \leq C \|\mathring{A}\|_{L^2}.$$

Estimate also for conformal factor. $+ \|u\|_{L^\infty(S^2)}$

on LHS, $f^* \delta_{\mathbb{R}^n} = e^{2u} g_{S^2}$.

Applications: Foliations of Riem mflds by special surfaces (codim 1).

Remk $W(f) = \frac{1}{4} \int \|H\|^2 d\mu \geq 4\pi$, $f: \Sigma^2 \to \mathbb{R}^n$.

$W(f) = \frac{1}{2} \int |\mathring{A}|^2 + 2\pi \chi(\Sigma)$.

$\Rightarrow \chi(\Sigma) = 2$.

Def (I) Σ Riem surface, $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ is called conf. immersion $f \in W^{2,2}_{conf}(\Sigma, \mathbb{R}^n)$ if

$g_{ij} = e^{2u} \delta_{ij}$ in loc. conf. coordinates, $u \in L^\infty$.

Rmk. $f \in W^{2,2}_{conf} \Rightarrow u \in \frac{1}{2}\log\left(\frac{1}{2}|\nabla f|^2\right) \in W^{1,2}$.

(II) f branched conf. if $f \in W^{2,2}_{conf,loc}(\Sigma \setminus S, \mathbb{R}^n)$, $|S| < +\infty$

$$\int_{\Sigma \setminus S} (1+|A|^2) < \infty.$$

Results. (Kuwert-Li '12):

① $\exists m \in \mathbb{N}$ $u(z) = (m-1)\log|z| + \text{GOOD}$.

② $f_k \in W^{2,2}_{conf}(\Sigma, \mathbb{R}^n)$, $W(f_k) \le C$.

$\Rightarrow \exists\ \sigma_k : \mathbb{R}^n \to \mathbb{R}^n$ möbius s.t.

$$\sigma_k \circ f_k \to f \in W^{2,2}_{conf,Br}(\Sigma, \mathbb{R}^n).$$

Weakly in $W^{2,2}_{loc}(\Sigma \setminus S, \mathbb{R}^n)$.

REGIDITY: (L-M-NGUYEN '14).

(I) $n=3$: ⓐ inverted Catenoid. $f_{CAT} \in W^{2,2}_{conf}(S^2, \mathbb{R}^3)$

$$\exists!\ x \in f_{CAT}(S^2),\ f^{-1}(x) = \{P_1, P_2\}.$$

$$\int |A|^2 = 24\pi,\quad W(f_{CAT}) = 8\pi.$$

Thm. $\exists \delta_0 > 0$ $\forall 0 < \delta < \delta_0$. $\forall f \in W^{2,2}_{conf}(S^2, \mathbb{R}^3)$.
with at least one double pt. and

$$24\pi \le \int |A|^2 \le 24\pi + \delta.$$

$\exists C(\delta)\ (\to 0$ as $\delta \to 0)$. $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ Möbius,
$\varphi : S^2 \to S^2$. repar. s.t. $\|\sigma \circ f \circ \varphi - f_{CAT}\|_{W^{2,2}} \le C(\delta)$.

- But not geometric instabilities, they are simply coordinate instabilities.

- Need Lyapunov functionals:
  Max principles fail b/c you throw away highest order terms.

  Lyapunov - keep them, get extra help from integration by parts.

Future: • Should be able to prove Ricci neckpinches are rot. symm.
  • Same ideas to MCF. $M^n_t \hookrightarrow \mathbb{R}^{n+1}$, $n \geq 3$.