The Cauchy Problem in GR

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1. Formulate Einstein Eq^{n}s.
2. Historical overview of development in GR.
3. —— " —— Cauchy problem.
4. Formulation of the IVP.
5. Non-linear hyperbolic PDE.
6. Global uniqueness.
7. Future global non-linear stability.

Einstein's eq^{n}s

- Classical: laws of physics invariant under Galilean transfms. (But not electromagnetism which lead to...)

- Special relativity: should be Lorentz transformations.
- Geometry: Minkowski space - Lorentz transfms are the isometries. $(\mathbb{R}^4, g_{min})$.

$$g_{min} = -dx^0 \otimes dx^0 + \sum_{i=1}^{3} dx^i \otimes dx^i$$

$\rightsquigarrow$ Maxwell's eq^{n}s are ok, but not gravity.
Need to modify gravity:
For Einstein;

- Equivalent principle: Inert mass = gravitational mass.
  Similarly: accerleration = gravitation.
  → Gravitation ⟺ distortion of geometry.

→ We need eq^n for a Lorentz metric g on a mfld M.

Classical case: $\Delta\varphi = c_0\rho$. φ gravitational potential
ρ - matter density
c_0 - universal cnst.

Make eq^n to resemble this;

Rough structure: second order diff op. on g = matter sources.

- Need this to be indep of coordinates to need this to be a tensor field on LHS and RHS.

- Need a simple object to include matter density, current etc., etc, to RHS: stress energy tensor T; symmetric, covariant, 2-tensor, and conservation of energy ⇒ $\operatorname{div} T \equiv 0$.

LHS: Symmetric, covariant, 2-tensor, div free, 2nd derivatives.

→ LHS = $G + \Lambda g$, Λ - cosmological constant.

$$G = \underset{\text{Ricci}}{\underset{\uparrow}{Ric}} - \frac{1}{2}\underset{\text{Scalar}}{\underset{\uparrow}{S}}g$$

$$\boxed{\text{Einstein's } Eq^n s: \quad G + \Lambda g = T}$$

History:

- Beginning: finding explicit sol^{n}s.
  - 1916 Schwarzschild.
  - 1917 Einstein Universe.
  - 20's, 30's. FLRW.
  - 1963. Kerr metric.

- Are two sol^{n}s diff. coord reps of the same sol^{n}s?
  - singularities appear.
  - Understanding singularities is problematic.
    Including Einstein thought that $r = 2m$ is a real singularity in Schwarzschild.
    It wasn't until the 1950's this was understood to be a failure of coords.

  $\Rightarrow$ It's necessary to take a geometric, coord indep perspective.

Geometry: 50's & 60's: Kruskal plane, singularity th^{m}s (Hawking & Penrose), Lorentz geometry.

Hawking & Penrose: singularity = causal geodesic incompleteness.

Null geodesic $\langle \dot\gamma, \dot\gamma \rangle = 0$. Timelike (free falling particle). $\langle \dot\gamma, \dot\gamma \rangle < 0$.

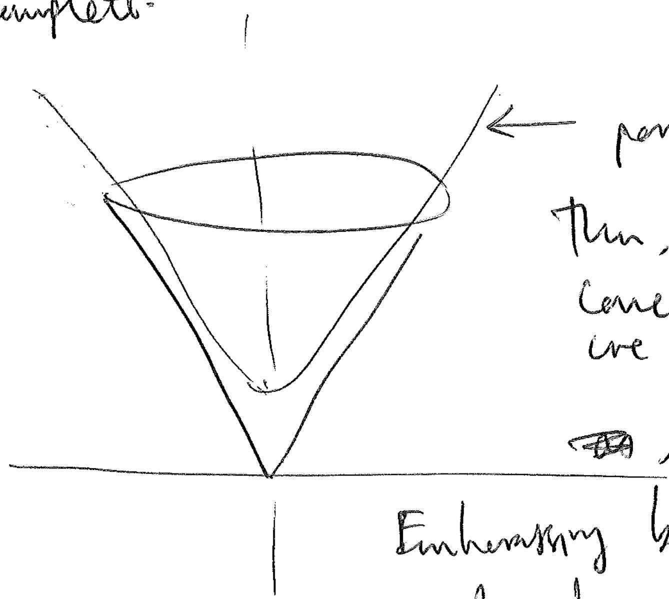
Causal geodesic incomp: $\exists\, \gamma$ timelike (i.e. particle) s.t. γ leaves spacetime in finite proper time.

Rmk. Mfld is maximal (ie not "incomplete").

So, $\exists$ spacetimes (M,g):

1. Hawking's thm applies: $\exists$ singularities.
2. (M,g) maximal. (Cauchy development - later on).
3. $Riem(g) = 0$.
4. (M,g) can be extended to be geodesically complete.

Take



Then, horizon is light cone and causal geodesics are incomplete.

light cone is "singularity".

Embarrassing because nothing violent happens at singularity.

Cauchy Problem:

- 1916 - Einstein: the gravitational field propagates at a speed bounded by light.

  criticism: special coordinates to make this work. Why is this invariant? (Particularly Eddington).
- 1921/22: de Donder, Lanczos: if coordinates are such that $g^{\alpha\beta}(\partial_\gamma g_{\alpha\beta} - 2\partial_\beta g_{\gamma\alpha}) = 0$, then

$$Ric = 0 \Leftrightarrow \underbrace{g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = h_{\mu\nu}}_{\text{Hyperbolic PDE}}.$$

· Kurt Stellmacher (1937): geometric (local) uniqueness result. (justifies coordinates). (Eddington's objection down the toilet).

· 1952: Yvonne Choquet-Bruhat: given i.d. satisfying the constraint eq^{s}, there is a corresponding sol^{n} to Einstein's equations!

↓ Sets the stage for Cauchy Problem.

Questions:
- Stability: perturb ~~flat~~ initial data $\Rightarrow$ sol^{n}'s perturb similarly.
- Asymptotics: a) singularities b) expanding direction
  c) AF $\mathcal{I}^+$ (radiation far out & near ∞).
  d)
- SCC: strong cosmic censorship.
  determinism is lost via IVP. Ie, $\exists$ sol^{n}'s for initial data. Ie, maximal Cauchy development can be extended in equivalent ways. So, can't determine which spacetime you're in.

The Cauchy Problem

Ex. Consider (M, g) where $M = I \times \mathbb{R}^3$,

$$g = -dt^2 + a^2(t)\,\bar{g}.$$

Want sol^{n} to Einstein's Eq^{n}: matter = dust., ie $T = \rho\, dt^2$.

$\underline{\text{Eq}^s}$. (I) $\quad 3\left(\frac{\dot a}{a}\right)^2 = \rho$, $\quad (G_{00} = T_{00})$.

(II). $\quad 0 = 0$, $\quad (G_{0i} = T_{0i})$.

(III). $\quad 2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 = 0 \quad (G_{ij} = T_{ij})$.

(IV) $\quad \dot\rho + 3\frac{\dot a}{a}\rho = 0. \quad (\text{div } T = 0)$.

__Procedure__: 1) Choose i.d. $\{a(t_0), \dot a(t_0), \rho(t_0).\}$ so that (I) holds.

2) Choose a & ρ - soln^{ns} to (3) & (4).

3) Hope for the best... (we only know (I) initially need to know in evolution.

__Define__: $\quad f = 3\left(\frac{\dot a}{a}\right)^2 - \rho$.

compute $\dot f = \cdot$ (using (III) & (IV)) $= -3\frac{\dot a}{a}f$.

$+ \quad f(t_0) = 0 \Rightarrow f(t) = 0 \ \forall t$.

__Note__: ① Broken diffeomorphism invariance by insisting on structure for g via this choice of coordinates.

② System of evolution equations & constraints.

__General Case__: Consider $R_{\alpha\beta} = 0$.

In local coords:

(V) $\quad R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)} + \underbrace{\Gamma\Gamma + \Gamma\Gamma + \Gamma\Gamma}_{\text{quad terms in Christ symbols.}}$

Here, $\Gamma^{\alpha} = g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}$, $\Gamma_\alpha = g_{\alpha\beta}\Gamma^{\beta}$.

$$\nabla_\mu \Gamma_\nu = \partial_\mu \Gamma_\nu + \Gamma^{\alpha}_{\mu\nu}\Gamma_\alpha .$$

$$\nabla_{(\mu}\Gamma_{\nu)} = \tfrac{1}{2}\left(\nabla_\mu \Gamma_\nu + \nabla_\nu \Gamma_\mu\right).$$

trouble term.

<u>Gauge source functions.</u> – need to break diffeo. inv. to get wave.

Idea: replace Γ_ν in ~~term~~ second term with F_ν.
(dependent on coords & metric but <u>no derivatives.</u>)

<u>Define</u> (6) $\hat{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{(\mu}D_{\nu)}$, where
$$D_\nu = F_\nu - \Gamma_\nu .$$

<u>Remark.</u> Γ_ν ~~not~~ are <u>not</u> a tensor field, but by correct choice of F_ν, can ensure D_ν <u>is</u>.

Then, $\hat{R}_{\mu\nu} = 0$ — hyperbolic system of PDEs.

Assume $\hat{R}_{\mu\nu} = 0$, then (6) gives $R_{\mu\nu} = -\nabla_{(\mu}D_{\nu)}$.

$$\Rightarrow \boxed{G_{\mu\nu} = -\nabla_{(\mu}D_{\nu)} + \tfrac{1}{2}\left(\nabla^{\gamma}D_{\gamma}\right) g_{\mu\nu}.}$$

Bianchi identity $\Rightarrow$ divergence RHS $= 0$.

$$\boxed{\nabla^\mu \nabla_\mu D_\nu + R_\nu{}^\mu D_\mu = 0.}$$ ← Important note: homogeneous wave eq^{n}.

As in the case $f = 3\left(\frac{\dot{a}}{a}\right)^2 - \rho$ initially 0, the Discrepancy D_ν initially $0 \Rightarrow D_\nu = 0$ for all time. So, get sol^{n} to $R_{\mu\nu} = 0$.

Ex. $\Box u = 0$ I.d. $u(t_0, \cdot)$, $u_t(t_0, \cdot)$.

$\{t = 0\}$ - Cauchy hypersurface, spacelike.
↗ timelike curves ~~are along transversal~~. that are extendible, intersect exactly once.

In our case: $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = F_{\mu\nu}(g, \partial g)$. (✱✱)

Data should be on a spacelike hypersurface w.r.t. g
Data $g_{\alpha\beta}|_\Sigma$, & normal derivative to. $g^{\alpha\beta}|_\Sigma$.

Geometric data: Induced metric & 2nd f.f.
(of the metric) . But don't really give data for
(Additional freedom).

How should the remaining data be specified?

Use requirement $D_\mu = 0$ in Σ.

Once you've done this right → no remaining freedom for (✱✱).

Rmk $D_\mu = 0$ means that you can choose coords. so that $F_\nu = \Gamma^\mu_\nu$. I.e, gauge function can be chosen to match connected Christ.

How to ensure $\nabla_\mu D_\nu = 0$?

If (M, g) is such that $Ric = 0$, and Σ spacelike hypersurface, then.

$$G(N, N) = 0, \quad G(N, X) = 0.$$

N normal to Σ $\qquad$ X tangential to Σ

$$(***) \quad \begin{cases} \bar{R} - \bar{k}_{ij}\bar{k}^{ij} + (\operatorname{tr} k)^2 = 0. \\ \bar{\nabla}^j \bar{k}_{ji} - \nabla_i \operatorname{tr} k = 0. \end{cases}$$

So if we want to solve vacuum eq^{n}s, necessary to satisfy.

Solving the vacuum eq^{n}s.

i) Let $(\Sigma, \bar{g}, \bar{k})$ be vacuum initial data.

I.e, Σ - n-mfld, $\bar{g}$ - Riem. on Σ,

$\bar{k}$ - ~~[illegible]~~. symm. covariant 2-tensor. field on Σ.

$(\bar{g}, \bar{k})$ satisfies $(***)$.

2) $(\bar{x}, \mathcal{U})$ local coords on $\Sigma \rightsquigarrow (x, \mathbb{R}\times\mathcal{U})$.
local coords on $\mathbb{R}\times\mathcal{U}$. $x = (x^0, x^1, \dots, x^n)$.

3) Choose gauge source functions on $\mathbb{R}\times\mathcal{U}$.

4) Choose initial data $g_{\mu\nu}|_{t=0}$, $\partial_t g_{\mu\nu}|_{t=0}$.
$(g_{00}|_0 = -1,\ g_{0j}|_0 = 0)$.
so that $D_\mu|_{t=0} = 0$, and which induce $\bar{g}$ and $\bar{k}$.

5) Solve $\hat{R}_{\mu\nu} = 0$ in a nbh of $\{0\}\times\mathcal{U}$.

6) Since $G_{\mu\nu} = -\nabla_{(\mu} D_{\nu)} + \frac{1}{2}(\nabla^\sigma D_\sigma) g_{\mu\nu}$.
and $D_\alpha|_{t=0} = 0$, and constraint eq^{n}s hold.
$\Rightarrow \nabla_\alpha D_\beta|_{t=0} = 0$.

[Constraint eqns $\Rightarrow$ contraction $G_{\mu\nu}$ with time & normal is 0. & $g_{\mu\nu}$ disappears.

~~contraction of~~ and also $0 = \nabla_{(\mu} D_{\nu)} = 0$.].

Since D satisfies homo wave $\Rightarrow \nabla_\alpha D_\beta = 0\ \forall t$.

7) Patch together to get global sol^n.

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Local existence:

<u>Thm</u>. Let $(\Sigma, \bar{g}, \bar{k})$ be vacuum i.d. (satisfy vacuum constraint eq^{ns}). Then $\exists$ a sol^n (M, g) to Einstein's vacuum equations and an embedding $i: \Sigma \to M$. s.t.

(2) $i^* g = \bar{g}$, $i^* k = \bar{k}$, k induced 2nd ff. on $i(\Sigma) \hookrightarrow M$ by g.

Moreover, $i(\Sigma)$ is a Cauchy hypersurface in (M, g).

(M, g) - <u>globally hyperbolic</u> development of $(\Sigma, \bar{g}, \bar{k})$. (Because $i(\Sigma)$ Cauchy).

<u>Problem</u>: There are infinitely many such developments, and some which are incompatible (2). But uniqueness can be obtained if one asks for a <u>globally hyperbolic</u> maximal development.

necessary, otherwise uniqueness is untrue.

Non-linear hyperbolic PDEs.

<u>ODE's</u>: Consider

$$(1) \begin{cases} \frac{dx}{dt}(t) = f(t, x(t)). \\ x(t_0) = x_0. \end{cases}$$

f continuous in a nbh of (t_0, x_0), then $\exists$ solution. (typically non-unique).

If f is continuous and Lipschitz w.r.t. 2nd argument, then:

- If x_1, x_2 are sol^{n}s. on $I_1, I_2 \overset{\text{open}}{\subset} \mathbb{R}$, $t_0 \in I_1 \cap I_2 \neq \emptyset$.

  then $x_1(t) = x_2(t)$ $\forall t \in I_1 \cap I_2$.

  $\rightsquigarrow$ Maximal existence interval. $I_{max} = (T_{min}, T_{max})$.

  $I_{max} = \bigcup_\alpha I_\alpha$ (all intervals of existence $t_0 \in I_\alpha$).

Pf. of local existence:

(1) Setup iteration: $x_0(t) = x_0$, $x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s))\,ds$.

(2) Prove convergence in $C_b(I, \mathbb{R}^d)$.

Rmk In more general settings, replace $\mathbb{R}^d$ by $X \leftarrow$ some appropriate Banach space.

Global existence: Either $T_{max} = \infty$ or $\lim_{t \to T_{max}^-} |x(t)| = \infty$.

This is a continuation criterion which is very important in non-linear PDE.

"the solution can be continued as long as it remains bounded."

Ex. Consider $\ddot{x} + x^3 = 0$, with $x_1 = x_2$, $x_2 = \dot{x}$;

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix}.$$

Moreover, $\partial_t \left(\frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4\right) = \{\text{exercise}\} = 0$.

So, $\frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$ remains bounded $\Rightarrow$ global existence $(T_{min}, T_{max}) = (-\infty, +\infty)$.

Further: Get cont. criteria. for weak $|x(t)|_{\mathcal{N}}$ $\rightsquigarrow$ low reg.

Bootstrap arguments.

Consider $\ddot{x} + 2\alpha\dot{x} + \beta^2 x = f(x,\dot{x})$

where $\alpha, \beta > 0$, $\alpha^2 < \beta^2$ and

$$|f(x,\dot{x})| \leq C\,(|x|^2 + |\dot{x}|^2).$$

1) Do small perturbations of zero initial data yield future global sol$^{\text{ns}}$.

2) What are the future asymptotics.

$\rightarrow$ use <u>energy</u> methods.

<u>Energy:</u> Consider $\ddot{x} + 2\alpha\dot{x} + \beta^2 x = 0$, and

$$E = \tfrac{1}{2}\left(\dot{x}^2 + 2\alpha\, x\dot{x} + \beta^2 x^2\right).$$

Then, 1) $\frac{dE}{dt} = -2\alpha E$, (decays exponentially).

2). $|2\alpha\, x\dot{x}| \leq \frac{\alpha}{\beta}\cdot 2|\beta x \dot{x}| \leq \frac{\alpha}{\beta}(\dot{x}^2 + \beta x^2)$.

$\Rightarrow$ E pos. def.

$$\Rightarrow \quad \tfrac{1}{2}\left(1 - \alpha/\beta\right)(\dot{x}^2 + \beta^2 x^2) \leq E \leq \tfrac{1}{2}\left(1 + \tfrac{\alpha}{\beta}\right)(\dot{x}^2 + \beta^2 x^2)$$

$\Rightarrow$ $\exists c_1, c_2 > 0$.

$$c_1(\dot{x}^2 + x^2) \leq E \leq c_2(\dot{x}^2 + x^2).$$

$$c_i = c_i(\alpha,\beta).$$

<u>Note:</u> (1) + (2) $\Rightarrow$ $E(t) = e^{-2\alpha t}E(0)$.

and $|\dot{x}(t)| + |x(t)| \leq C e^{-\alpha t}$. $\forall t \geq 0$.

For the non-linear eq^{n}:

$$\frac{dE}{dt} = -2\alpha E + (\dot{x} + \alpha x) f ,$$

However, $|\dot{x} + \alpha x| \leq C (\dot{x}^2 + x^2)^{\frac{1}{2}} \leq C E^{\frac{1}{2}}$.

and $|f| \leq c (\dot{x}^2 + x^2) \leq C E$.

$$\Rightarrow \quad \frac{dE}{dt} \leq -2\alpha E + C E^{3/2} .$$

Letting $E_\alpha(t) = e^{2\alpha t} E(t)$,

$$\frac{dE_\alpha}{dt} \leq C e^{-\alpha t} E_\alpha^{3/2}(t) . \qquad (2)$$

The main ingredients of a bootstrap argument:

1) Bootstrap assumptions: An assumption on the solution in $[0,T]$. Here:
   a) Sol^{n}s exist on $[0,T]$
   b) $E_\alpha^{\frac{1}{2}}(t) \leq \varepsilon$ on $[0,T]$. (ε to be determined later.)

2). An assumption concerning the initial data. Here: $E_\alpha^{\frac{1}{2}}(0) \leq C_0 \varepsilon$.

3). Prove that $1) + 2) \Rightarrow$ 1) can be extended a little more than T, i.e., $[0, T+\delta]$.

$\Rightarrow$ $T = \infty$.

If $0 \le t \le T$, then $E_\alpha(t) \overset{(2)}{\le} E_\alpha(0) + \int_0^t C e^{-\alpha s} E_\alpha^{3/2}(s)\,ds$

$$\overset{(1)+(2)}{\le} c_0^2 \varepsilon^2 + \int_0^t C e^{-\alpha s} \varepsilon^3\,ds$$

$$\le c_0^2 \varepsilon^2 + \frac{C}{\alpha}\varepsilon^3.$$

Choose $c_0 = \frac{1}{2}$, $\frac{4C}{\alpha}\varepsilon = 1 \Rightarrow E_\alpha(t) \le \frac{1}{2}\varepsilon^2$.

$\Rightarrow$ Set on which bootstrap assumptions exist is connected, closed and by: it is open $\Rightarrow$ D' $[0,T) = [0,\infty)$

Let A be the set of $t \ge 0$ s.t. the bootstrap assumption held on $[0,t]$. Then A is non-empty, closed and open. Thus sol^n exists globally to the future and $E_\alpha^{\frac{1}{2}}(t) \le \varepsilon \quad \forall t \ge 0$.

$$\Rightarrow |\dot{x}(t)| + |x(t)| \le C\varepsilon e^{-\alpha t} \quad \forall t \ge 0.$$ sol^n decays exponentially.

<u>Rmk</u> Continuation criteria gives global existence but <u>without</u> any control on the solution. The initial data can be large. But energy method gives decay information, cost: initial data has to be small.

Non-linear hyperbolic PDEs.

Want to solve:

$$(*) \qquad g^{\alpha\beta}(u)\,\partial_\alpha\partial_\beta u = F(u, \partial u).$$

Solve $(*)$ given $u(0,\cdot) = u_0$, $u_t(0,\cdot) = u_1$.

For simplicity,

$$(1) \quad \begin{cases} \Box u = F(u, \partial u). \\ u(0,\cdot) = u_0. \\ u_t(0,\cdot) = u_1. \end{cases}$$

Iterate:

Let $u_0(t,x) = u_0(x)$, solve

$$\begin{cases} \Box u_{n+1} = F(u_n, \partial u_n). \\ u_{n+1}(0,\cdot) = u_0. \\ \partial_t u_{n+1}(0,\cdot) = u_1 \end{cases}$$

Q. Does the sequence u_n converge? In what space?

C^k spaces? If the i.d. are in $C^{k+1} \times C^k \ni (u_0, u_1)$ the corresponding sol^n to $\Box u = 0$ is typically not C^{k+1}!

$(1+1)$ dim, this is ok. But $(d+1)$, $d \geq 2$, this is bad.

Real analyticity: $\to$ changing fnctn in nbhd changes everything $\to$ wrecks finite speed of prop, destroys causality.

Idea: Construct space from equation itself $\to$ Back to energy.

Energies

$\Box u = 0$. (say, just in $\mathbb{R}^n$).

Let $E = \frac{1}{2}\int_{\mathbb{R}^d} (|u_t|^2 + |\nabla u|^2)\, dx$.

Diff + int by parts $\Rightarrow$ $\partial_t E = 0$.

Demand that u_0, u_1 be s.t.:

$$\sum_{|\alpha| \le k+1} \|\partial^\alpha u_0\|_2 < \infty, \quad \sum_{|\alpha| \le k} \|\partial^\alpha u_1\|_2 < \infty \quad \text{Reading off from Energy.}$$

Natural function spaces: Sobolev spaces $H^k(\mathbb{R}^d) \ni u$.

$$\sum_{|\alpha| \le k} \int_{\mathbb{R}^d} |\partial^\alpha u(x)|^2\, dx < \infty.$$

Lot of hard work $\Rightarrow$ For I small enough, $0 \in I$,
$u_n \to u$ in the space

$$\underbrace{C_b^1}_{\text{time derivative}}(I, H^k(\mathbb{R}^d)) \cap C_b^0(I, H^{k+1}(\mathbb{R}^d)).$$

Note: This is for k large enough.

If $I_{max} = (T_{min}, T_{max})$ is the maximal interval of existence, then either:

(I) $T_{max} = +\infty$ or (II) $\|u(t,\cdot)\|_{C^1} + \|\partial_t u(t,\cdot)\|_{C^0}$ is unbounded on $[0, T_{max})$.

↑ Continuation criterion in this context

This is for a large class of equations. But maybe you can drop $\|\partial_t u(t,\cdot)\|_{C^0}$ term can be dropped for a particular eqn.

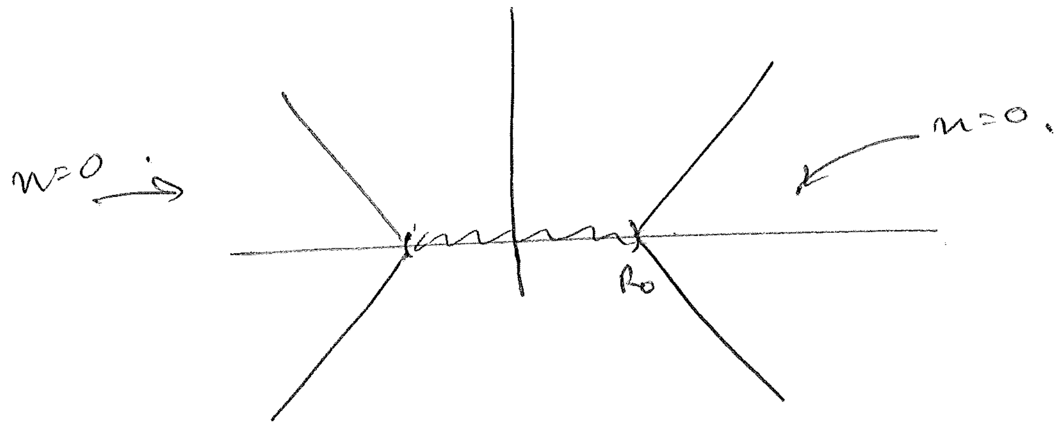
Ex. For $\Box u = F(u)$, it's sufficient to control $\|u(t,\cdot)\|_{C^0}$. Consider.

$$\begin{cases} u_{tt} - u_{xx} = -u^k & k \text{ odd}. \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases}$$

$f, g \in C_0^\infty(\mathbb{R})$, $f(x) = g(x) = 0 \quad \forall |x| \geq R_0$,

then $E = \int_{\mathbb{R}} (u_t^2 + u_x^2 + \frac{2}{k+1} u^{k+1})\, dx$. is conserved.

Moreover if $R(t) = R_0 + |t|$, $u(x,t) = 0$ ~~outside $R(t)$~~ for $|x| \geq R(t)$.



Estimate: $|u(t,x)| = \left| \int_0^x u_x(t,\xi)\, d\xi \right|$.

$$\leq \int_{-R(t)}^{R(t)} |u_x(t,\xi)|\, d\xi.$$

$$\leq \int_{-R(t)}^{R(t)} \frac{1}{2}(1 + u_x^2(t,\xi))\, d\xi.$$

$$\leq R(t) + \frac{1}{2}E.$$

grows linearly. conserved.

$\Downarrow$

cannot blow up in finite time $\Rightarrow$ global existence.

Rmk f, g do not even need to be cptly supported b/c of finite speed of prop. But, we don't know anything about the solution.

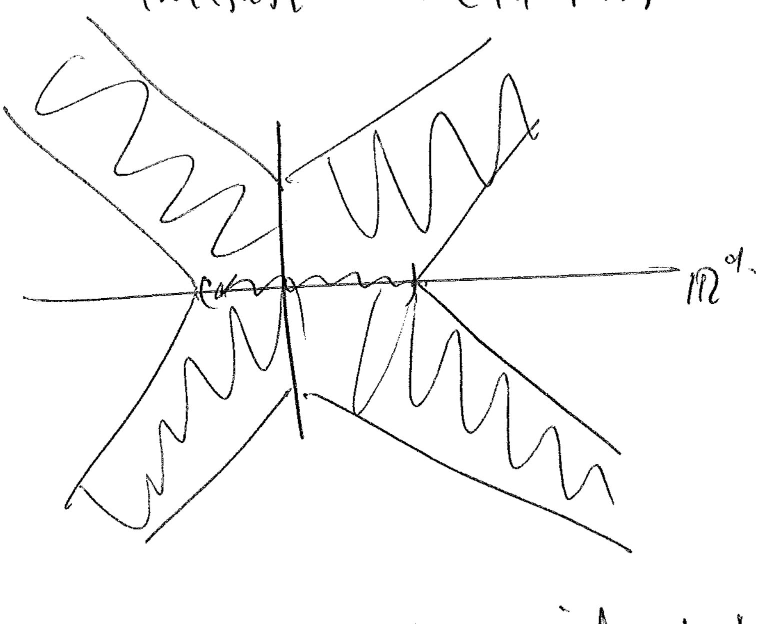
Stability. Consider $\Box u = F(u, \partial u)$ where
$$|F(u, \partial u)| \leq C(|u|^2 + |\partial u|^2).$$

Is $u = 0$ a stable sol$^{\underline{n}}$?

Consider $\begin{cases} \Box u = 0 & (d+1)\text{-dim}. \\ u(0,x) = f(x). & u_t(0,x) = g(x) \end{cases}$

where $f(x) = g(x) = 0$ for $|x| \geq R$.

1) If d is odd: $u(t,x) = 0$ if $||t| - |x|| \geq R$.
$$|u(t,x)| \leq C(1+|t|)^{-\frac{d-1}{2}}.$$



2) If d is even, $u(t,x) = 0$ if $|x| \geq |t| + R$ and.
$$|u(t,x)| \leq C(1+|t|)^{-\frac{d-1}{2}} \underbrace{(1+|t|-|x|)^{-\frac{d-1}{2}}}_{\text{if you're far from light cone, get extra decay.}} \leftarrow \text{not sure about this exp.}$$

Ex. If $d \geq 4$, and $F(u, \partial u) = G(\partial u)$, then, global existence holds for small data.

Stability is better in high dimensions because there are more directions to disperse in. But we're really interested in $d = 3$.

For instruction, look at:

$$\left.\begin{array}{ll}(2) & \Box u = -u_t^2 \\ (3) & \Box u = u_t^2 - |\nabla u|^2.\end{array}\right\} \text{toy models for Einstein}$$

with i.d. $u(0,x) = \varepsilon f(x)$, $u_t(0,x) = \varepsilon g(x)$.

$\exists$ constant $c > 0$ s.t. for small ε, $\exists$ smooth sol^n to (2) and ~~(3)~~ smooth sol^n to (3) on $T_\varepsilon = e^{c/\varepsilon}$ $\leftarrow$ almost global existence!

$$\left.\begin{array}{l}\text{(2) Blow up regardless of initial data} \\ \text{(3). Existence for small } \varepsilon.\end{array}\right\} \text{Why?}$$

Solution dissipates quicker along tangential direction to light cone, and are large in transverse directions.

(2) turns out to have a quadratic non-linearity in transverse direction, whereas (3) does <u>not</u>

Importance of null condition - (Clear conceptual reason).

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Maximal globally hyperbolic development.

Def. Given vacuum initial data $(\Sigma, \bar{g}, \bar{k})$, a maximal globally hyperbolic development (MGHD) is a globally hyperbolic development (M, g) with embedding $\iota : \Sigma \to M$ s.t. if (M', g') is any other glob. hyp. dev. then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image and s.t.

$$\psi^* g = g', \quad \psi \circ \iota' = \iota.$$

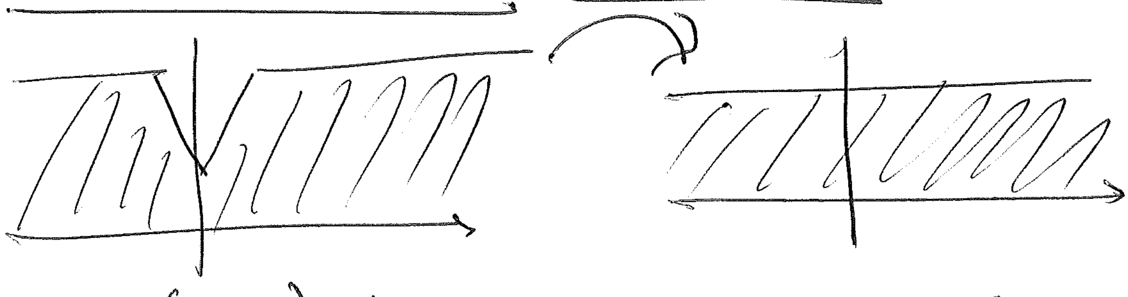
Remark. Stronger requirement than usual maximality, but this is automatically unique.

Thm. Given vacuum i.d., there is a unique MGHD.

Main analytical ingredients:

(I) Local existence.

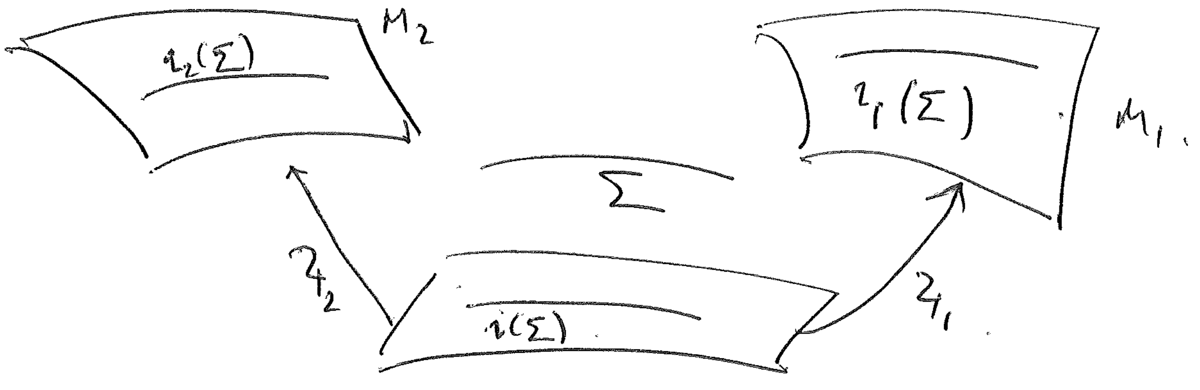
(II). Local uniqueness. Can't extend.



Thm. Let $(\Sigma, \bar{g}, \bar{k})$ be vacuum i.d. and let (M_j, g_j), $j = 1, 2$, be globally hyperbolic developments with embeddings $\iota_j : \Sigma \to M_j$. Then there is a GHD (M, g)

with embedding $i: \Sigma \to M$ and maps $\psi_j : M \to M_j$ s.t. ψ_j diffeomorphism onto their images and.

$\psi_j^* g_j = g_1$ & $\psi_j \circ i = i_j$.



<u>Idea of Pf</u>: 1) Construct appropriate local coordinate neighborhoods of points $i_j(\Sigma)$ with so that the corresponding contracted Christoffel symbols coincide

2) Since initial data are the same and PDE's are the same, the corresponding change of coordinates is a local isometry, preserving the i.d.

3) Patching together $\rightsquigarrow$ an isometry of a globally hyperbolic nbhd of $i_2(\Sigma)$.

<u>Existence of an MGHD</u>:

- Fix i.d. $(\Sigma, \bar{g}, k)$.
- Let $\mathcal{M}$ denote the isometry classes of globally hyperbolic developments (can argue this is a <u>set</u>!).
- $\mathcal{M}$ is partially ordered:

$[M_1, g_1] \leq [M_2, g_2]$ If $\exists$ isometric embedding $(M_1, g_1) \hookrightarrow (M_2, g_2)$ preserving inclusion of Σ

by $\psi: M_1 \to M_2$ $\quad \psi \circ \iota_1 = \iota_2$.

- Every totally ordered subset of $\mathcal{M}$ has upper bound:

$[M_\alpha, g_\alpha]$ totally ordered.

Then take union over α and start making identifications.
$\Rightarrow$ again unified with metric that solves vacuum Einstein.

Zorn's Lemma $\Rightarrow$ $\exists$ maximal element.

Let (M, g) be a maximal element. Let (N, h) be <u>any</u> GHD. Want to prove $[N, h] \leq [M, g]$.

Let $C(N, M)$ be the set of pairs (U_N, ψ) s.t.

a) U_N is open, containing $\iota_N(\Sigma)$ Cauchy hypersurface in (U_N, h).

b) $\psi: U_N \to M$ diffeo onto image s.t. $\psi \circ \iota_N = \iota_M$.

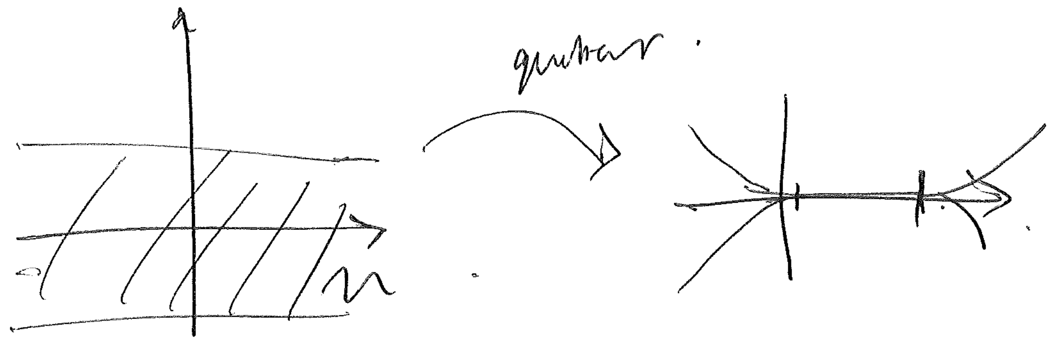
<u>Note:</u> • $C(N, M) \neq \emptyset$ by local uniqueness.

- If (U_i, ψ_i) are two elements of $C(N, M)$, then $\psi_1 = \psi_2$ on $U_1 \cap U_2$.

$\rightsquigarrow$ there is a partial ordering by set inclusion.
In fact, there is a unique maximal element (U, ψ).

<u>Proving $U = N$</u>: • Consider $N \amalg M$.
• Identify n with $\psi(n)$. $\rightsquigarrow$ equivalence relation.
$\rightsquigarrow$ quotient $\tilde{M}$.

$\tilde{M} \to$ might not be Hausdorff ?!!



But assume $\tilde{M}$ non-Hausdorff $\Rightarrow \mathcal{U}$ non-maximal.

$\Rightarrow$ $(\tilde{M}, \tilde{g})$ is a GHD extension (M, g). But (M, g) maximal $\Rightarrow$ $[\tilde{M}, \tilde{g}] \leq [M, g]$, $\tilde{M}$ can be embedded into M. By construction, N can be embedded into $\tilde{M} \Rightarrow$ $[N, h] \leq [M, g]$.

Pf of this: Assume $\tilde{M}$ non-Hausdorff. The non-Hausdorff points $\subseteq \partial \mathcal{U} \cup \partial \psi(\mathcal{U})$. If H is the set of non-Hausdorff points in $\partial \mathcal{U}$, there is a map $f: H \to \partial \psi(\mathcal{U})$.

Construct extension: • want to prove $\exists$ spacelike hypersurface, say $\bar{S}$ in N and a $p \in H$ s.t. $\bar{S} \setminus \{p\} \subset \mathcal{U}$.

• prove that $\psi(\bar{S} \setminus \{p\}) \cup \{f(p)\}$ is a spacelike hypersurface in M; and ψ extends to a C^∞ map from $\bar{S}$ to its image.

• Appeal to local uniqueness and thereby $\mathcal{U}$, can be extended.

Future global existence

Consider $g = -dt^2 + e^{2Ht}\bar{g}$ on $\mathbb{R} \times \mathbb{T}^n$;

$H > 0$ constant, $\bar{g}$ flat metric on $\mathbb{T}^n$.

Sol^n to vacuum Einstein with pos cosmological const.

$$G + \Lambda g = 0 \quad \text{where} \quad \Lambda = \frac{n(n-1)}{2} H^2.$$

Question: Is this sol^n future globally non-linearly stable? I.e., future global development stable, geodesically complete etc.?

Motivation: Cosmological "no hair" conjecture.

1) Choose an eq^n. Naive choice (I) $\hat{R}_{\mu\nu} = \frac{2}{n-1} \Lambda g_{\mu\nu}$.

where F_μ are contracted Christoffel symbols of the background.

Let (II) $F_\mu = n H g_{0\mu}$.

Here, use:

$$(3) \quad \hat{R}_{\mu\nu} - \frac{2}{n-1}\Lambda g_{\mu\nu} + M_{\mu\nu} = 0,$$

where $M_{00} = -2H g^{0\mu} D_\mu$, $M_{0i} = 2H D_i$, $M_{ij} = 0$

$$D_\mu = F_\mu - \Gamma_\mu.$$

- (3) is a perfectly good system of hyperbolic PDEs.

- If the i.d. satisfy the constraints, you can set up i.d. for (3) so that the resulting sol^{n}. to (3) solves $G+\Delta g=0$.

- Divide the terms involving at most first derivatives

"Irrelevant terms": those involving two factors of the type $\partial_t g_{ij}-2Hg_{ij}$, $g^{00}+1$, $g_{00}+1$, g_{0i}, g^{0i}, $\partial_0 g_{00}$, $\partial_0 g_{0i}$, $\partial_i g_{\alpha\beta}$.

"Relevant terms": Terms involving at most one such factor.

$\rightsquigarrow$ (4). $\hat{\square}_g n + (n+2)H\partial_0 n + 2n H^2 n + \Delta_{00} = 0$

irrelevant.

(5). $\hat{\square}_g u_m + H\partial_0 u_m + 2(n-2)H^2 u_m$
$- 2Hg^{ij}\Gamma_{imj} + \Delta_{0m} = 0.$

(6). $\hat{\square}_g h_{ij} + nH\partial_0 h_{ij} + \Delta_{ij} = 0$.

where $\Delta_{\alpha\beta}$ - irrelevant terms (quadratic, vanish on background))
$\hat{\square}_g := -g^{\alpha\beta}\partial_\alpha\partial_\beta$, $n = g_{00}+1$, $n_i = g_{0i}$, $h_{ij} = e^{-2Ht}g_{ij}$.

Ex. Replace $\hat{\square}_g$ with "$\partial_t^2 - e^{-2Ht}\Delta$", consider

$$\partial_t^2 v - e^{-2Ht}\Delta v + \alpha H\partial_t v + \beta H^2 v = F.$$

Where $\alpha>0$, $\beta\geq 0$ are constants.

Consider: $\mathcal{E}_{\gamma,\delta} = \frac{1}{2}\int_{\mathbb{T}^n} [v_t^2 + e^{-2Ht}|\nabla v|^2 + 2\gamma H v v_t + \delta H^2 v^2]\, dx$

If $\beta = 0$, choose $\gamma = \delta = 0$, if $\beta > 0$, choose

$\gamma = \frac{\alpha}{2}$, $\delta = \beta + \frac{\alpha^2}{2}$.

If $\beta > 0$, $\exists\, C > 1$:

$$\underbrace{\frac{1}{C}\int_{\mathbb{T}^n} (v_t^2 + e^{-2Ht}|\nabla v|^2 + H^2 v^2]\, dx}_{} \leq \mathcal{E}_{\gamma,\delta} \leq C$$

Regardless of whether $\beta = 0$ or not, $\exists\, \eta > 0$:

$$\partial_t \mathcal{E}_{\gamma,\delta} \leq -\eta\, \mathcal{E}_{\gamma,\delta} + \int_{\mathbb{T}^n} (v_t + \alpha v) F\, dx.$$

a) $F = 0$ & $\beta > 0$ $\rightsquigarrow$ Exp. decay + Sob embedding.

$\Rightarrow$ $\|v_t(t,\cdot)\|_{C^k} + \|v(t,\cdot)\|_{C^k} \leq C_k\, e^{-aHt}$.

b) $F = 0$, $\beta = 0$; then $\|v_t(t,\cdot)\|_{C^k} \leq C_k e^{-aHt}$.

$\Rightarrow$ $\exists\, v_\infty \in C^\infty(\mathbb{T}^n)$: $\|v(t,\cdot) - v_\infty\|_{C^k} \lesssim e^{-aHt}$.

Similar estimates can be derived for

$$\hat{\Box}_g v + \alpha H v_t + \beta H^2 v = F.$$

In fact, let

$$\mathcal{E}_{\gamma,\delta}[v] = \frac{1}{2}\int_{\mathbb{T}^n} [-g^{00} v_t^2 + g^{ij}\partial_i v\, \partial_j v - 2\gamma H g^{0\mathrm{o}} v v_t + \delta H^2 v]\, dx$$

Let $E_{\gamma,\delta,k} = \sum_{|\alpha| \leq k} \mathcal{E}_{\gamma,\delta}[\partial^\alpha v]$.

Then, $\exists \eta > 0$: $\frac{dE_{\gamma,\delta,k}}{dt} \lesssim -\eta H E_{\gamma,\delta,k} + \sum_{|\alpha| \leq k} \int_{\mathbb{T}^n} \Big[(\partial^\alpha \partial_t v + \gamma H \partial^\alpha v)$

$$\times \Big(\partial^\alpha F + [\hat{\Box}_g, \partial^\alpha] v \Big) \Big\} + \underbrace{\Delta_{E;\gamma,\delta}[\partial^\alpha v]}_{\text{Error terms.}} \Big] dx$$

$+ 2\Delta$.

<u>Need to estimate:</u> I) $\|\Delta_{\alpha\beta}{}^i\|_{H^k}$. II) $\| [\hat{\Box}_g, \partial^\alpha] v \|_{L^2}$ $|\alpha| \leq k$.
(III). $\| \Delta_{E,\gamma,\delta} [\partial^\alpha v] \|_{L^2}$. $|\alpha| \leq k$.

Bootstrap assumptions.

1) Assump. concerning metric in C^0:

$\|g_{00} + 1\|_{C^0}$ small, $\| e^{-2Ht - 2k} g_{ij} - \delta_{ij} \|_{C^0}$ small

$\underbrace{g_{00}}_{-ve}$ positive here.

$\| e^{Ht} g_{0i} \|_{C^0}$ small.

2). Define energies associated with.

a) $u \rightsquigarrow \hat{H}_{\rho,k}$.
b) $u_m \rightsquigarrow \hat{H}_{S,k}$.
c) $h_{ij} \rightsquigarrow \hat{H}_{m,k}$.

Let $\hat{H}_k = \hat{H}_{\rho,k} + \hat{H}_{S,k} + \hat{H}_{m,k}$.

<u>Bootstrap ass.</u> $\hat{H}_{k_0} \leq \varepsilon$.

Means to improve bootstrap assumptions:

Prove: (I) $$\frac{d\hat{H}_{ep,k}}{dt} \leq -aH\hat{H}_{ep,k} + CH\varepsilon e^{-aHt}\hat{H}_k^{\frac{1}{2}}\hat{H}_{ep,k}^{\frac{1}{2}}.$$

(II) $$\frac{d\hat{H}_{s,k}}{dt} \leq -aH\hat{H}_{s,k} + CH\hat{H}_{m,k}^{\frac{1}{2}}\hat{H}_{s,k}^{\frac{1}{2}} + CH\varepsilon e^{-aHt}\hat{H}_k^{\frac{1}{2}}\hat{H}_{s,t}^{\frac{1}{2}}.$$

lose.

(III) $$\frac{d\hat{H}_{m,k}}{dt} \leq He^{-aHt}\hat{H}_{m,k} + CH\varepsilon e^{-aHt}\hat{H}_k^{\frac{1}{2}}\hat{H}_{m,k}^{\frac{1}{2}}.$$

Adding these: you lose!

But this is a system of eq^ns. → improve bootstrap for ~~each~~ (III) and (II) and plug improved estimate into .

Idea: Improve bootstrap for some components before using that for others.