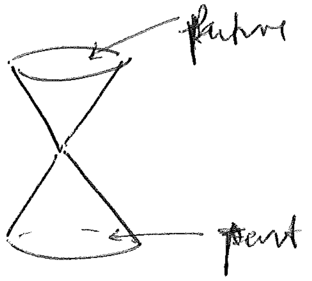
On the Geometry & Topology of initial data sets in GR - G. Galloway. 15/09/2015.

(Accompanied by ~~Beam~~ lecture notes & ESI off website).

- Time-orientability:



assignment can be made ctsly.

$\Longleftrightarrow \quad \exists T \in C^\infty(TM), \qquad g(T,T) < 0$ (timelike)

- Timelike vector or null vector $\Longleftrightarrow$ "causal" vector (Def^n of causal).

  Causal X, Y, $\quad |X+Y| \geq |X| + |Y| \leftarrow$ Twin paradox.

- causal curve γ : $\quad \dot{\gamma}(t)$ causal.

- Causal relations $\begin{cases} p \ll q : & \text{timelike curve from } p \text{ to } q. \\ p < q : & \text{causal curve from } p \text{ to } q. \end{cases}$

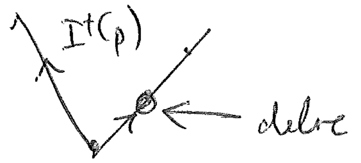
- Notation. $I^+(p) = \{q \in M : p \ll q\}$ timelike future.

  $J^+(p) = \{q \in M : p < q\}$ causal future.

$J^+(p) = I^+(p) \cup \partial I^+(p)$ for Minkowski space.

- Rmk Topology & geometrically can drastically affect causal structure. See flat spacetime cylinder.

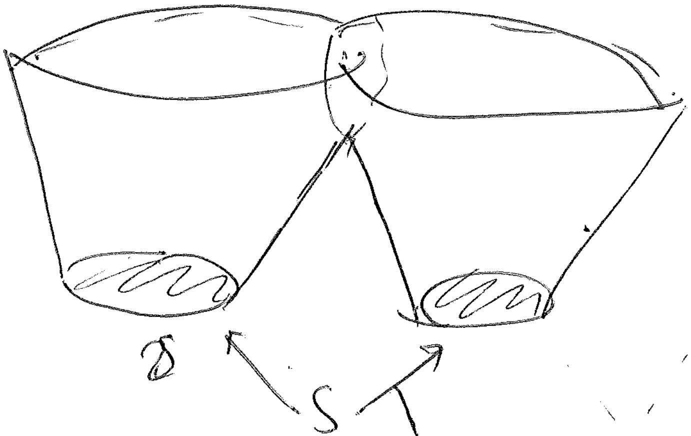
$I^+(p)$ open, but $J^+(p)$ does not need to be closed.



• Achronal $\partial I^+(S)$ for some $S \subset M$.

Achronal: $S \subset M$ if no two points $p \in S$, $q \in S$ can be joined by timelike curve.

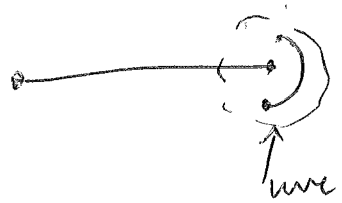
$\partial I^+(S)$



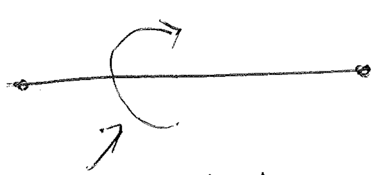
Compared to:

• Edge point $p \in \bar{S}$, S achronal, $\forall$ nbh $U \ni p$, $\exists$ timelike curve from $I^-(p,U)$ to $I^+(p,U)$.

M^2:



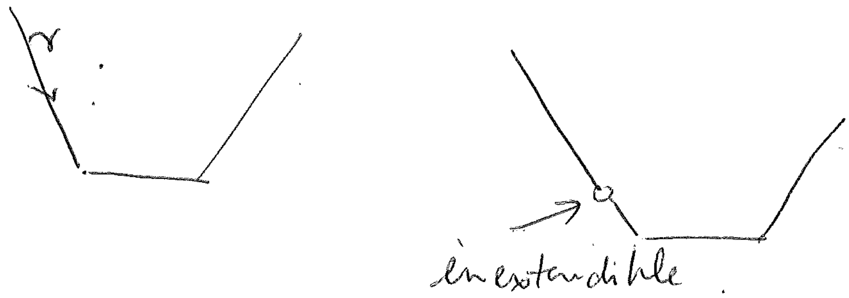
M^3:



$\Rightarrow$ Edge set has something to do with dimension.

• Inextendible. $\lim_{t\to b} \gamma(t)$ does not exist.

for $\gamma:(a,b)\to M$.



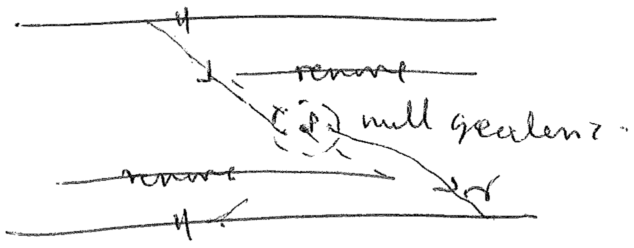
• ++Chronology condition++: there are no timelike curves that are closed.

++Thm++. Every compact spacetime contains closed timelike curves.

→ why there is limited interest in compact spacetimes in GR.

++Causality++: there are no closed ~~timelike~~ causal curves.

• "Almost closed".

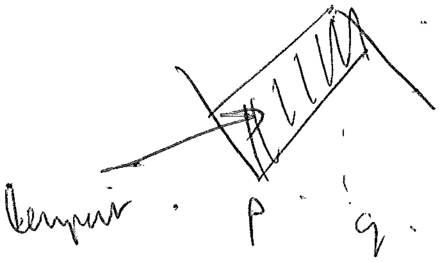


} cylinder to one pt.

• ++Strong causality++: at $p \in M$ $\exists$ has arbitrarily small causally convex nbhds.

• Strong causal ~~spacetime~~ compact K. Then, $\gamma : [0,b) \to M$ future extendible causal eventually leaves K if $\gamma(0) \in K$.

• M: globally hyperbolic if strongly causal and $J^+(p) \cap J^-(q)$ are compact $\forall p,q \in M$.



E.g. Minkowski $\setminus \{0\}$ is not globally hyp.

globally hyperbolic $\Rightarrow$ no naked singularities for Einstein eq^{n}s.

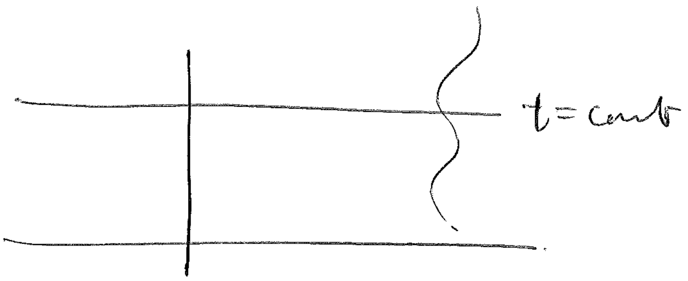
$\Downarrow$

$J^\pm(A)$ closed A compact.

$J^+(A) \cap J^-(B)$ cpct A, B cpct.

• Cauchy surface: $S \subset M$ achronal subset met by every inextendible causal curve.

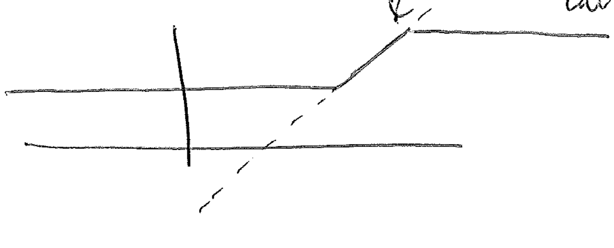
Ex. M^2

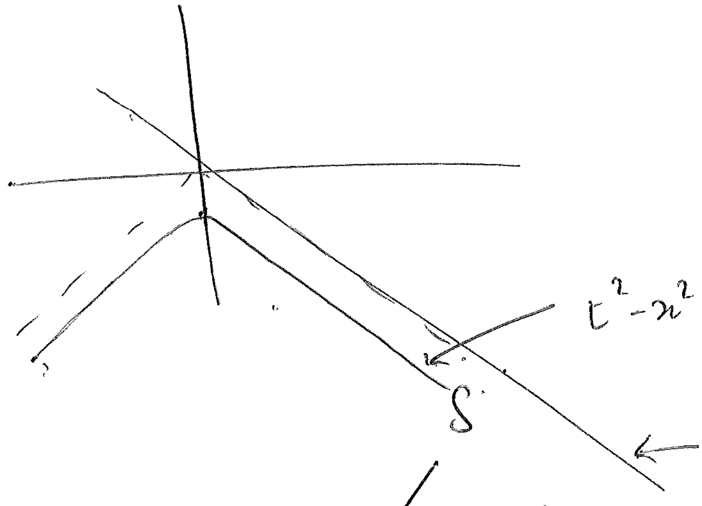


Also:

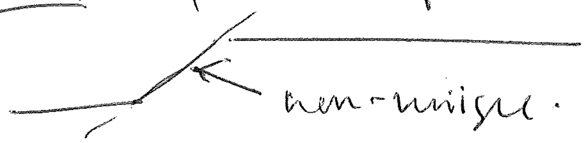
null piece

causal curve, not intersecting in a unique way.





S Cauchy $\Rightarrow$ (I) $\partial I^+(S) = S = \partial I^-(S)$.
(II) closed C^0 hypersurface.
(III). Inextendible timelike curves intersect exactly once.
Rmk Prev example with.



Th^m
Every globally hyp. spacetime contains a Cauchy S.

Introduce measure $\mu(M) = 1$ and. $f(p) := \dfrac{\mu(J^-(p))}{\mu(J^+(p))}$.

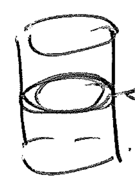
f cts and strictly increasing, called time function.
(cts: interval compactness; strictly increasing: strong causality.)

Slices $S = \{p \in M : f(p) = 1\}$. is Cauchy.
In fact, this gives a foliation C^0 of M.

• M globally hyperbolic.

$\Rightarrow$ $M \cong S \times \mathbb{R}$ and any two Cauchy S and S' are <u>homeomorphic</u>.

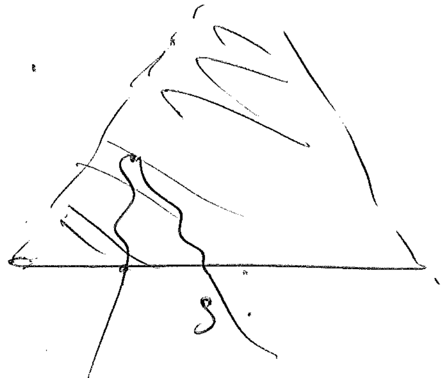
• Compact C^0 achronal $S \Rightarrow$ <u>Cauchy</u>.

(This $\not\Rightarrow$ M cpct, think  $\leftarrow S$.

<u>Domain of dep:</u> S achronal $\subset M$.

$D^+(S) = \{p \in M$: any past inextendible causal curve meets $S\}$.

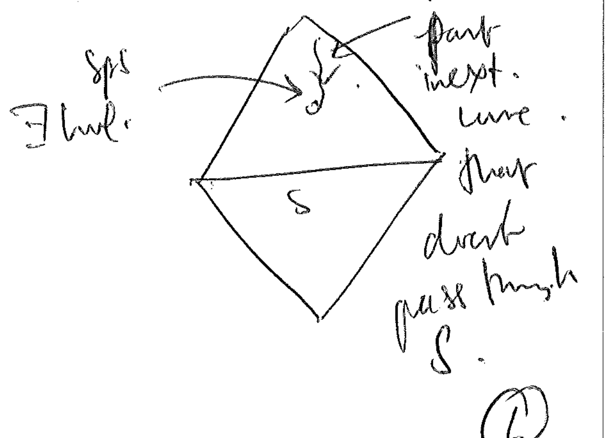
$D(S) = D^+(S) \cup D^-(S)$.



• S achronal; (I) S Cauchy iff $D(S) = M$.

(II) $\mathrm{int}(S)$ has strong causality + internal spacetime.

Says that Cauchy surfaces are the right stuff for initial value problem.

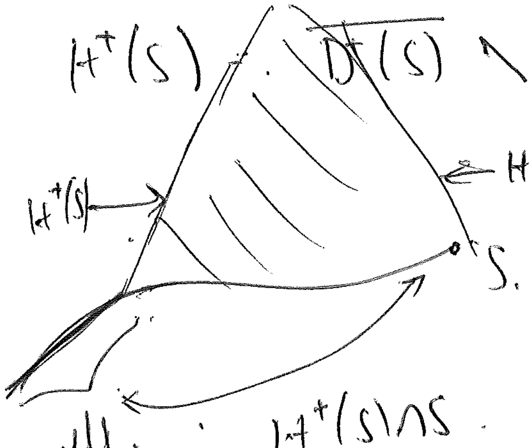


Actually: S cauchy iff M ~~hyp~~ globally hyp.

~~S achronal~~

- Cauchy horizon: $S \subset M$ achronal, $H^+(S)$ is the ~~$\partial D^+(S)$~~ future boundary.

$H^+(S) := \overline{D^+}(S) \setminus I^-(D^+(S))$.



$H^+(S)$ achronal, $\partial D^+(S) = H^+(S) \cup S$.

Null hypersurfaces (next time, preclude).

- Tangential to the null cone, event horizon of Schwarzschild & Kerr are null hyp.

Einstein: $R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}$.

$T_{ij} = 0 \iff R_{ij} = 0$ (Vacuum Einstein).

Null Energy Condition (NEC): $T(X,X) \geq 0$ X null.

Dom —— // —— (DEC): $T(X,Y) \geq 0$ X, Y causal.

Geometry of Null hypersurfaces.

17/09/2015.

Null hypersurface in (M, g) is $S \subset M$ submanifold s.t. $g(p)=0$ on $T_pS \times T_pS$.

$\exists$ direction of degeneracy k_p: $\langle k_p, X\rangle = 0 \ \forall X \in T_pS$.

(1) k null vector, $\langle k_p, k_p\rangle = 0$ and.

(2) $[k_p]^\perp = T_pS$.

(3) If X is not a scalar multiple of k_p.
$\Rightarrow$ X spacelike.

$\Rightarrow$ $\exists$ null vectorfield k.

Ex ① null hyperplane in $\mathbb{M}^{n+1}$ (minkowski).
② Null cones. $\partial I^-(p)$ and $\partial I^+(p)$.

- Study Shape of null hypersurface. But k is orthogonal to S $\rightsquigarrow$ analogue of shape operator in Riem geom.

Work "mod k". $X \sim Y$ if $X - Y = \lambda k$, $\exists \lambda$.

$$\left.\begin{aligned} T_pS/k(p) &= T_pS/\sim \\ TS/k &= TS/\sim \end{aligned}\right\} \Rightarrow \begin{aligned} &h([X],[Y]) = g(X,Y). \\ &\text{is a Reim metric} \\ &\text{on } TS/k. \end{aligned}$$

Null Weingarten map: $b = b_k : T_pS/k \to T_pS/k$.

$$b([X]) = [\nabla_X k]$$

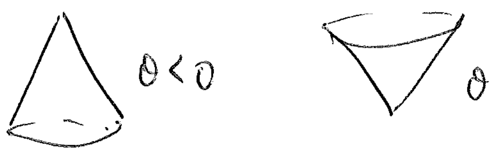
& b is self-adjoint!

Null 2nd ff: $B([X],[Y]) = h(b([X]),[Y])$.

Null mean curvature: $\theta = \operatorname{tr} b$, scalar.

~~With~~ Let $\{e_1, \dots, e_{n-1}\}$ o.n. basis for T_pS/k.

Then, $\theta = \operatorname{tr} b = \operatorname{div} k$. "measuring expansion of future development of null ~~vector~~ generators towards future."



$\theta < 0$ $\theta > 0$.

Sign of θ invariant under scaling of k by factors.

<u>Comparison Theorem.</u>

$\eta : I \to M$ future directed <u>affinely paramet.</u> null geodesic. Fix $s \in I$, $b = b(s)$ based at $\eta(s)$ w.r.t. $k = \eta'(s)$.

$$b(s) = b_{\eta'(s)} : T_{\eta(s)}S/\eta'(s) \to T_{\eta(s)}S/\eta'(s).$$

Gives Riccati $b' + b^2 + R = 0$. $b' = \nabla_{\eta'(s)} b$.

$$\theta' = -\operatorname{Ric}(\eta', \eta') - \sigma^2 - \left(\frac{1}{n-1}\right)\theta^2. \qquad \sigma = (\operatorname{tr} \hat{b}^2)^{\frac{1}{2}}$$

"shear scalar"

(Raychaudhuri equation).

Raychaudhuri $\Rightarrow$.

M spacetime satisfying $R(X;X) \geq 0$, S C^∞ null hyp.

Null generators of S are future geodesically complete.

$\Rightarrow \theta \geq 0$.

Rmk Einstein $Ric - \frac{1}{2}Rg = 8\pi T$.

(NEC) is $T(X,X) \geq 0$ for all null X.

But null $X \Rightarrow g(X,X) = 0$.

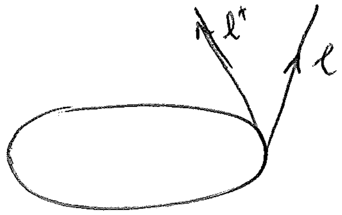
So, $Ric(X,X) = 8\pi T(X,X)$.

and $Ric(X,X) \geq 0$ iff $T(X,X) \geq 0$.

$Ric(X,X) \geq 0$ is the more important (NEC) if we don't assume Einstein eq^{n}s.

Hyp $\Rightarrow$ cross sections of S are nondecreasing in area moving to the future.

Penrose Singularity Thm



Assoc. 2nd f. forms:

$$\chi_\pm(X,Y) = g(\nabla_X l^\pm, Y).$$

Σ

$$\theta^\pm = tr_\gamma \chi_\pm = div_\Sigma l_\pm.$$

Σ round sphere Euclidean in Minkowski $\Rightarrow \theta_- < 0, \theta_+ > 0$.

Σ Trapped: Gravity is strong: $\theta_- < 0$, $\theta_+ < 0$.

Singularity Thm: under several circumstances, $\exists$ incomplete timelike or null geodesics.

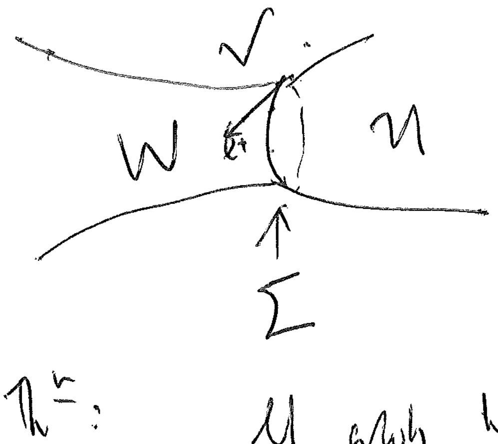
Incompleteness $\Rightarrow$ spacetime comes to an end in past or future. $\rightsquigarrow$ assoc. with gravitational collapse.

Penrose: grav. field becomes sufficiently strong, trapped surfaces appear, development of singularities inevitable.

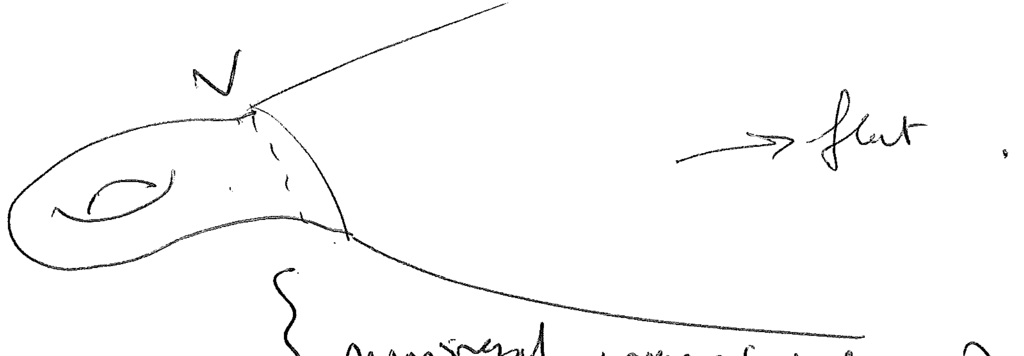
<u>Thm</u>. M globally hyperbolic spacetime satisfying null energy cond. with a noncpt Cauchy surface. M contains trapped $\Sigma \Rightarrow M$ is future null geo. incomp.

This Thm has <u>outer trapped</u> $(\theta_+ < 0)$ variant.

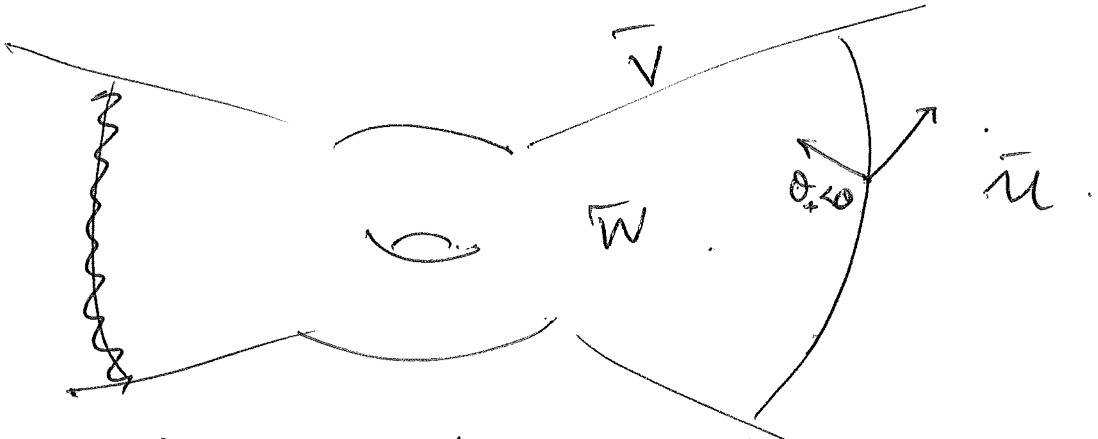
<u>Thm</u>: M globally hyp, NEC, V C^∞ Cauchy hyper. $\Sigma \subset V$ hyp. smooth, cpt w/o bdry, separates V into inside U & outside W, $\bar{W}$ noncpt. Σ outer trapped
$\Rightarrow$ M future null geodesically incomplete.



<u>Topological Thm</u>: M glob. hyper., NEC, asymp. flat Cauchy hyp. V. $\pi_1(V) \neq 0 \Rightarrow M$ future null incomplete.
$\rightsquigarrow$ "Topological censorship"



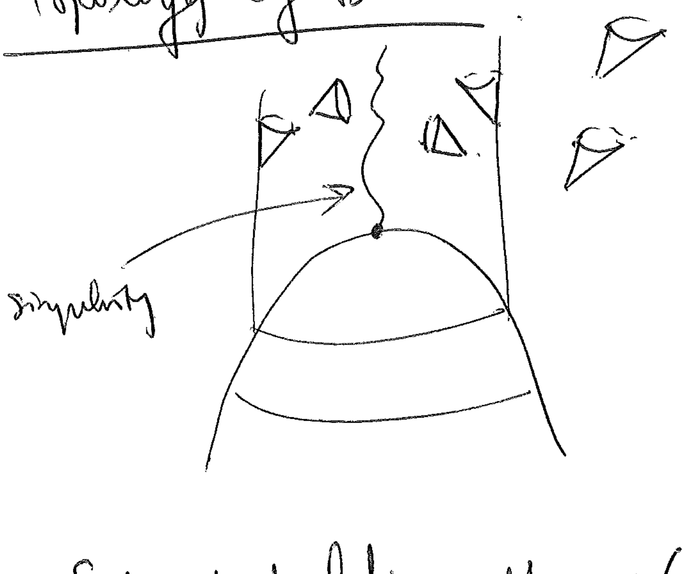
universal cover (at least 2 ends).



$\bar W$ noncpt because of presence of more than one end.

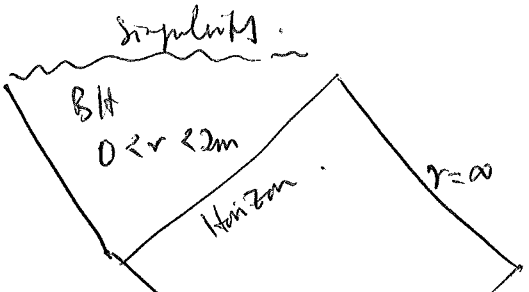
large ball.

Topology of BHs.



Schwarzschild: $$g = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

↑ static, non-rot.



Kerr sol^n for time-indep, rotating.

Physical significance: It's believed BHs settle down to Kerr eventually.

Th^m (Hawking BH top. th^m)

(M, g) $(3+1)$ spacetime, asymp. flat, DEC.
Cross sections Σ event horizon are top 2-spheres.

Higher dim. BHs.

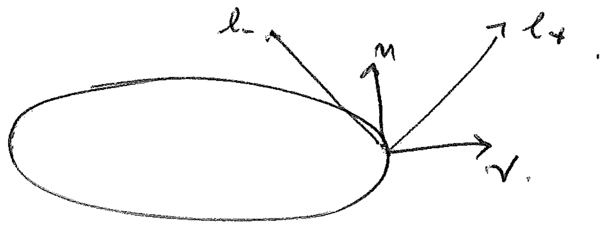
- 2002: In $(4+1)$ dim, event horizon, BH horizon,
  top $= S^2 \times S^1$ and so uniqueness breaks down.

Marginally outer trapped

Motivation: Hawking's Th^m in higher dim.

Initial data set in M^{n+1} spacetime is (V^n, h, k).
V^n spacelike, h metric (Riem), k 2nd ff.

Σ^{n-1}. 2-sided hyp:



$$\left.\begin{aligned} l_+ &= n+\nu \text{ f.d. outward } \Sigma^{n-1} \\ l_- &= n-\nu \text{ f.d. inward} \end{aligned}\right\} \Rightarrow \begin{aligned} \chi_\pm(X,Y) &= g(\nabla_X l_\pm, Y) \\ \theta_\pm &= \operatorname{div}_\Sigma l_\pm \\ &= \operatorname{tr}_\Sigma k \pm H. \end{aligned}$$

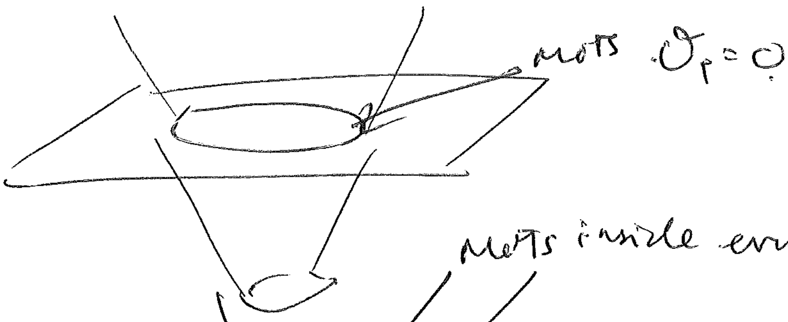
$\Rightarrow$ time-symmetric. $k = 0$ and $\theta_+ = H$.

Σ trapped: $\theta_\pm < 0$.

Σ outer trapped: $\theta_+ < 0$

Σ marginally outer trapped (MOTS): $\theta_+ = 0$.

Stationary BHs:



Dynamical BH: