Foliations in asymptotically flat 3-mflds. 14/09/2015.

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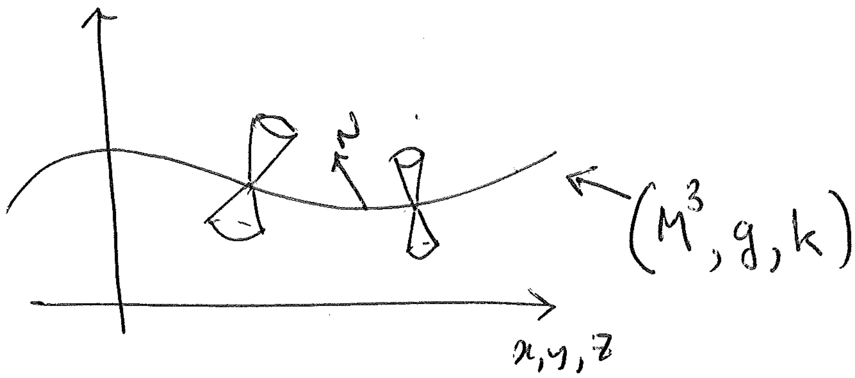
Let $(L^4, h, -+++)$. Lorentzian 4-mfld.

isolated gravitating system ; Stars, black holes, ...
- can be isolated from the rest of the universe.

(*) $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}h_{\mu\nu} = 8\pi T_{\mu\nu}$. ⟵ Einstein field equations.

$\underbrace{\nabla^\mu T_{\mu\nu}}_{\text{divergence}} = 0$ ⟵ conservation of energy.

physics: "weak energy condition" : $T_{\mu\nu}X^\mu X^\nu \geq 0 \quad \forall X$ timelike



g induced metric.

$g = h|_{TM^3}$

k - second fundamental form.

g spacelike : g is Reimmanian!

Rmk. N is not at 90° to hypersurface as you would expect → Feature of problem.

Gauss-eqn's. Let $\{e_1, e_2, e_3\}$ be an o.n. frame for g. $e_0 = N$ future directed unit normal.

$$R_{ijkl} = \bar{R}_{ijkl} + k_{il}k_{jk} - k_{ik}k_{jl}. \qquad 1 \le i,j,k,l \le 3.$$

$$R_{ik} = \bar{R}_{ik} + \bar{R}_{i0k0} + k_{il}k^{l}_{k} - (\operatorname{tr} k)k_{ik}.$$

$$\Rightarrow \boxed{R = \bar{R} + 2\bar{R}_{00} + |k|^2 - (\operatorname{tr} k)^2}.$$

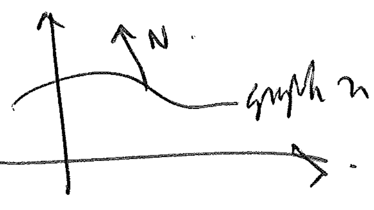
$$(*) \Rightarrow R = 16\pi T_{00} + |k|^2 - (\operatorname{tr} k)^2 \underset{\text{weak energy cond.}}{\ge} |k|^2 - (\operatorname{tr} k)^2 = |k|^2.$$

$\boxed{\text{Def}^{\text{n}} \text{ "Maximal hypersurface".}} \longrightarrow \boxed{\text{if } (\operatorname{tr} k) = 0.}$

Prop If $(M^3, g, k) \hookrightarrow (L^4, h)$ is a maximal hypersurface and (L^4, h) satisfies the weak energy condition, then $R(g) \ge 0$ on M^3.

Maximal hypersurfaces. $(\operatorname{tr} k) \equiv 0$. (Why is this a good assumption?)

Eg $M^3 = \text{graph } u \subset (\mathbb{R}^{3,1}, \eta)$, $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.



$$N = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}} \leftarrow \text{spacelike } |Du| < 1.$$

This is space like iff $|Du| < 1$.

$g_{ij} = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$, $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^{3,1}$ $x \longmapsto (x, u(x))$

$= \delta_{ij} - D_i u D_j u$.

$H = \mathrm{div}(N) = D_i \left(\frac{D_i u}{\sqrt{1-|Du|^2}} \right)$

$= \frac{1}{\sqrt{1-|Du|^2}} \left(\delta_{ij} + \frac{D_i u D_j u}{1-|Du|^2} \right) D_i D_j u \overset{!}{\equiv} 0$

quasilinear elliptic eq^n, uniformly elliptic $\Leftrightarrow$ $|Du| \leq 1-\delta$
$\delta > 0$..

<u>known</u>: (I) Dirichlet problem can be solved for bdry data φ on $\partial\Omega \subset \mathbb{R}^3$ if $|\varphi(x) - \varphi(y)| < |x-y|$.
(Bartnik - Simon)

(II) If $M^3 \subset \mathbb{R}^{3,1}$, $M^3 =$ graph u, $u: \mathbb{R}^3 \rightarrow \mathbb{R}$, $|Du| < 1$,
$H(\text{graph } u) \equiv 0$ $\Rightarrow$ $M^3 =$ spacelike plane in $\mathbb{R}^{3,1}$
("Bernstein th^m", Cheng-Yau).

(III) <u>R. Bartnik</u>: If $(L^4, h, -+++)$ has a time ~~like~~ function $t: L^4 \rightarrow \mathbb{R}$ with reasonable behaviour, asymptotic to Minkowski space in appropriate way; then, for $\forall t_0$ at spacelike ∞, $\exists!$ maximal surface in (L^4, h) with $t|_{M^3} \rightarrow t_0$ near spacelike ∞

Rmk on ~~(III)~~: Some loose ends + open problems.

I.e., ~~due~~ smoothness needed in (III).

Also, (II) if $H \equiv c \neq 0$, then problems.

Q. perturbations of Schwarzschild → does then fit the picture? Schwarzschild in ~~at~~ included in (III).

Study asymptotically flat 3-mflds with non-neg $R(g) \geq 0$.

Defⁿ (Asymp. flat): $M^3 \setminus \text{cpct} \simeq \mathbb{R}^3 \setminus B_{R_0}(x)$.

(possibly disjoint union of finitely many)

together with $x: M^3 \setminus \text{cpct} \to \mathbb{R}^3 \setminus B_{R_0}(x)$ s.t.

$$g_{ij}(x) = \delta_{ij} + P_{ij} \quad \text{with} \quad P_{ij} = O\left(\frac{1}{r^{\frac{1}{2}+\varepsilon}}\right) \quad r = |x|.$$

$$|\nabla^l P| = O\left(\frac{1}{r^{\frac{1}{2}+l+\varepsilon}}\right), \quad 1 \leq l \leq k.$$

(eg $k = 2$).

~~Examples~~

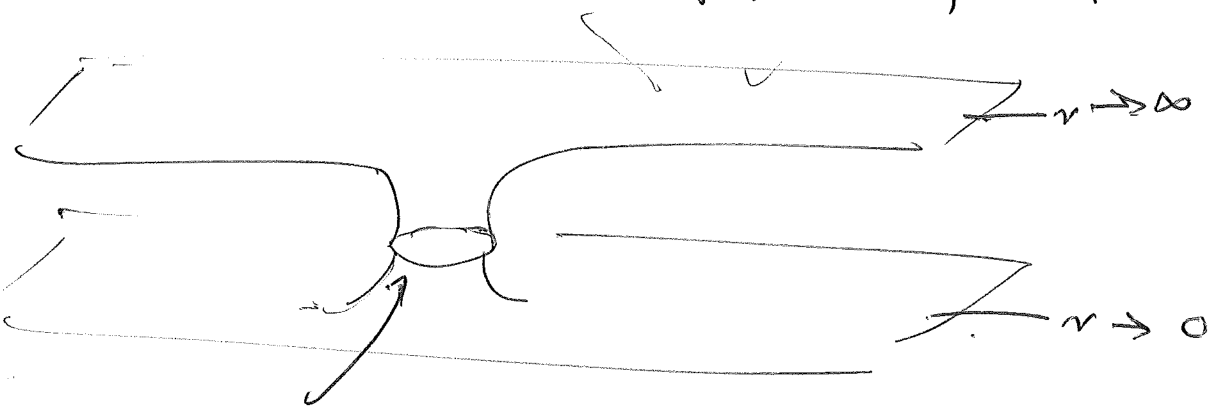
Rmk: Shouldn't expect $g_{ij} = \delta_{ij}$, because g_{ij} is a ~~pot~~ substitute for Newtonian potential. So, you expect decay $\sim \frac{1}{r^{\frac{1}{2}+\varepsilon}}$.

Examples: (I) $(\mathbb{R}^3, \delta)$

(II) spacelike Schwarzschild $g_{ij} = \delta_{ij}\left(1 + \frac{m}{2r}\right)^4$ $m > 0$. ← isotropic coordinates.

$\left[ds^2 = -\left(1 - \frac{2m}{\tilde{r}}\right) dt^2 + \left(1 - \frac{2m}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 \, d\Omega \ldots \right]$.

$M^3 = \mathbb{R}^3 \setminus 0$, 2 ends asymp. flat, reflection symm.

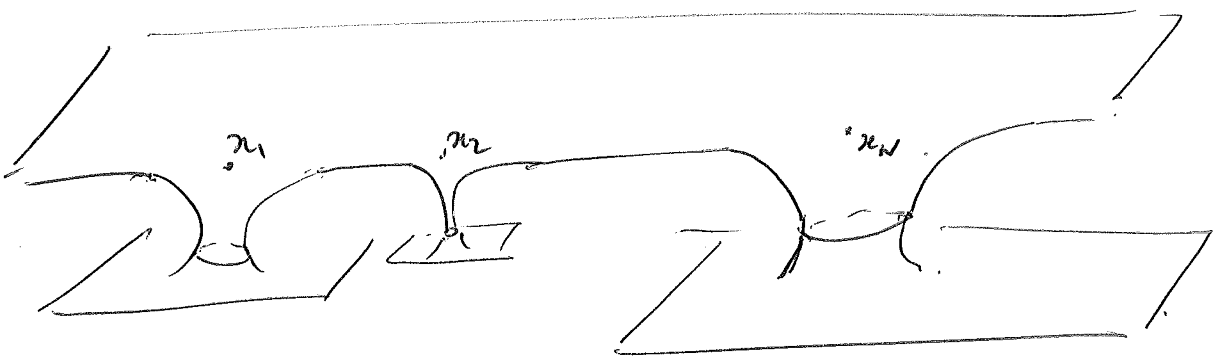


"Horizon" $\{r = \frac{m}{2}\} = \{\bar{r} = 2m\}$.

– $R(g) \equiv 0$, static black hole, exterior of static star.

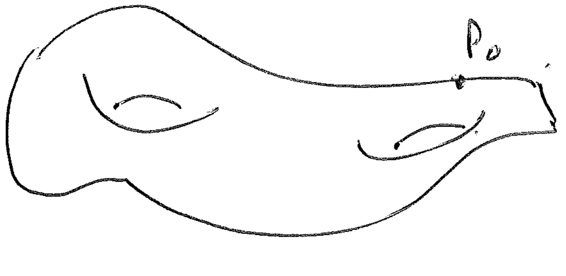
(III) $g_{ij} = \delta_{ij}\left(1 + \sum_{\ell=1}^{N} \frac{m_i}{|x - x_i|}\right)^4$; $R(g) \equiv 0$, $x_i \in \mathbb{R}^3$.

since g_{ij} in (II) satisfies Laplace eq^n. $M^3 = \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$



(IV) (R. Schoen). Let (M^3, g) closed (cpt, no bdry).

$R(g) \geq 0$.



$P_0 \in M^3$. Solve $-\Delta_g G + \frac{1}{8} R_g G = 4\pi \delta_{P_0}$.

Dirac mass at P_0.

G - Conformal Greens function. of g in p_0.

Can be solved with. $0 < G$ on $M^3 \setminus \{p_0\}$. s.t.

$$G(q) \sim \frac{1}{d_g(p,q)} + A + O(d_g(p_0,q)). \text{ near } p_0.$$

then consider: $\tilde{g}(p) := G^4(p) \cdot g(p)$, $\forall q \in M^3 \setminus \{p_0\}$.

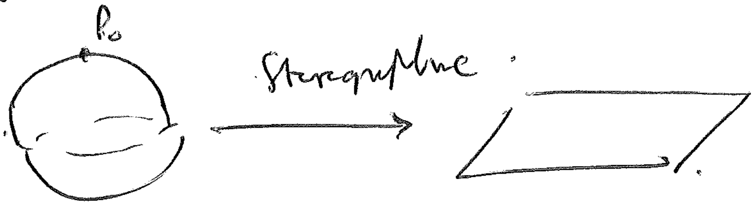
and $R(\tilde{g}) \equiv 0$.

If $\{x\}$ are normal coordinates near p_0 and $y = \frac{Ax}{|x|^2}$.

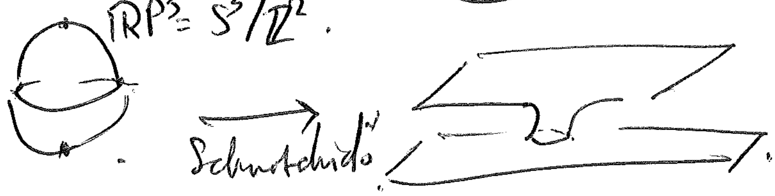
Then, $\tilde{g}_{ij}(y) = \delta_{ij} + \frac{2A}{|y|} + \text{l.o.}$

is, asymptotically flat.

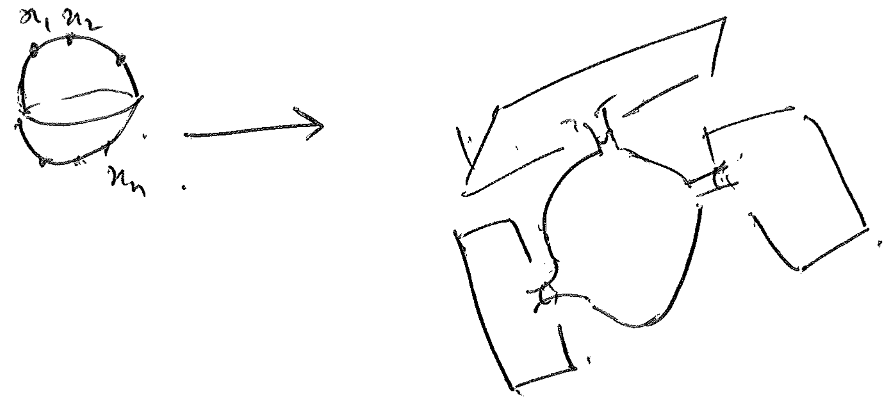
Revisit (I), (II), (III): (I).



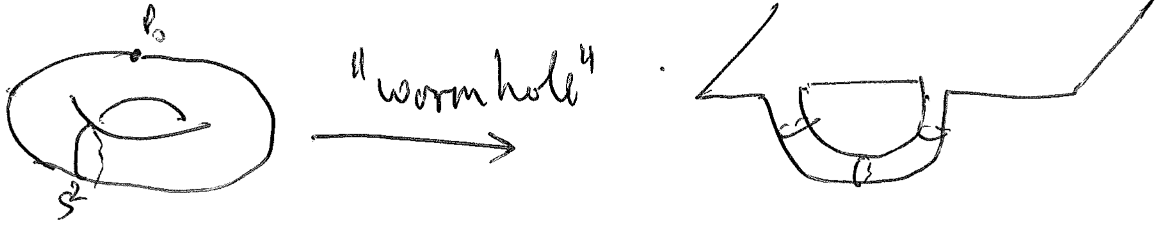
(II).



(III)



(IV). $(M^3, g) = (S^2 \times S^1, g_{star})$.



In Schwarzschild, $g_{ij} = \delta_{ij}\left(1+\frac{m}{2r}\right)^4$; $m>0$.

"mass" of the system (M^3, g).

If you take timelike geodesics and consider Kepler orbits, then, m plays the role of mass.

Since $R(g) \equiv 0 \Rightarrow$ no matter. So this is "mass" without having matter.

In general, ADM (M^3, g) asymptotically flat

$$m_{ADM} := \frac{1}{16\pi} \int_{S^2_\infty = \partial B_\infty} (g_{ij,i} - g_{ii,j})\, \nu^j \cdot d\sigma .$$

Check: $m_{ADM} = m$ in Schwarzschild.

$\rightsquigarrow$ A is the "mass" of $\tilde{g}$ as defined above.

Positive mass th$^{\underline{m}}$ (Schoen-Yau '79, Witten) : if (M^3, g) has $R(g) \geq 0$, asymp. flat $\Rightarrow m_{ADM}(M^3, g) \geq 0$.

"=" if $(M^3, g) = (\mathbb{R}^3, \delta)$.

Sol$^{\underline{n}}$ of Yamabe Problem (Schoen 1984). Came to this result implying $A \geq 0$.

From here on: Studying such asymptotically flat 3mflds by studying 2-hypersurfaces. Change of notation:

$(N^3, \bar{g})$ 3-mfld, asymp. flat. $F: M^2 \longrightarrow (N^3, \bar{g})$.

<u>Notation:</u> $g = \bar{g}\big|_{TM^2}$, $g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle_{\bar{g}}$.

2nd F.F., $A = \{h_{ij}\}$ in a frame $\{e_1, e_2, \nu\}$.

ν normal (vector).

$$-h_{ij}\nu^\alpha = \frac{\partial^2 F}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + {}^{\bar{g}}\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}.$$

$$h_{ij} = \langle \bar{\nabla}_{e_i}\nu, e_j\rangle = -\langle \bar{\nabla}_{e_i} e_j, \nu\rangle .$$

$$\frac{\partial \nu^\alpha}{\partial x^i} = h_{ij}\, g^{jl}\frac{\partial F}{\partial x^l} + \underbrace{\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\nu^\gamma}_{\text{term missing}}$$ (interpret curvature as change in normal).

Eigenvalues w.r.t. g: λ_1, λ_2:

$$H = g^{ij}h_{ij} = \lambda_1 + \lambda_2 , \quad |A|^2 = \lambda_1^2 + \lambda_2^2 = h^{ij}h_{ij}.$$

Gauss & Codazzi:

$$R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.$$

$$R_{ik} = \bar{R}_{ik} - \bar{R}_{i0k0} + H h_{ik} - h_{il}h^l_k.$$

$$R = \bar{R} - 2\bar{R}_{00} + H^2 - |A|^2.$$

$$\nabla_i h_{jk} - \nabla_j h_{ik} = \bar{R}_{0jik}.$$ Codazzi eq$^{\text{n}}$s.

Now consider families, i.e., $F: M^n \times [0,T) \to (N^{n+1}, g)$.

$\downarrow$

Th$^{\underline{m}}$. Sps F moves in normal direction with speed f.

$$\partial_t F(p,t) = f(p,t)\cdot \nu(p,t). \quad , p \in M^3, \ t \in [0,T).$$

Then: (I). $\partial_t g_{ij} = 2f\cdot h_{ij}$, $\quad \partial_t g^{ij} = -2f h^{ij}$.

$$\partial_t (d\mu) = Hf(d\mu).$$

(II). $\partial_t \nu = -\nabla f$

(III) $\partial_t h_{ij} = -\nabla_i \nabla_j f + f(h_{je}h^l_i - \bar{R}_{0i0j})$.

(IV). $\frac{d}{dt} H = \frac{d}{dt}(g^{ij}h_{ij}) = -\Delta f - f(|A|^2 + \bar{R}_{00})$

$$= L_{stab} f.$$

stability operator

Rmk. Main toolbox. Once you know them, choose favorite speed f: mean curvature, inverse mean curvature, etc.

Pf idea: (I) normal coordinates $\{x\}$ near $p \in M^3$ and normal coords $\{y\}$ near $F(p,t_0)$. ($\Rightarrow 0 = {}^{g(t)}\Gamma^i_{jk}(p) = \bar{\Gamma}^\alpha_{\beta\gamma}(F(p,t_0))$

but not outside $(p, t = t_0)$.)

$$\partial_t g_{ij} = \partial_t \langle \partial_{x^i} F, \partial_{x^j} F\rangle_{\bar{g}} = \langle \partial_{x^i}(f\cdot\nu), \tfrac{\partial F}{\partial x^j}\rangle + \langle \partial_{x^i}F, \partial_{x^j}(f\nu)\rangle_{\bar{g}}$$

$$= 2f h_{ij}.$$

(II). $\langle \nu,\nu\rangle \equiv 1$, $\quad 0 \equiv \langle \nu, \partial_{x^i}F\rangle$, $\langle \partial_t \nu, \nu\rangle \equiv 0$.

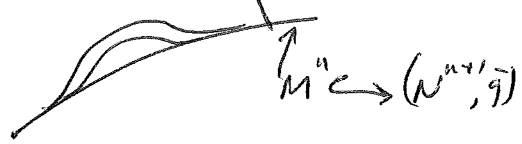
$$\Rightarrow \quad \langle \partial_t \nu, \partial_{x^i} F\rangle = -\langle \nu, \partial_{x^i}(f\cdot \nu)\rangle = -\nabla_i f.$$

(III). $\partial_t h_{ij} = \partial_t \left\langle -\frac{\partial^2 F}{\partial x^i \partial x^j} - \bar\Gamma^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}, \nu \right\rangle = \cdots$

<u>Note:</u> Diff $\bar\Gamma \rightsquigarrow$ curvature term.

<u>Corollary</u> Let $F_0 : M^n \hookrightarrow (N^{n+1}, \bar g)$ be a surface, and let f with cpt spt be a given speed for a variation.

$$F : M^n \times (-\varepsilon, \varepsilon) \to (N^{n+1}, \bar g)$$



<u>Then,</u> (I) $\delta(|M^n_t|, f) = \int_{M^n} fH \, d\mu$. $\quad \delta$ = first variation

(II) $\delta^2(|M^n_t|, f) = \int_{M^n} -\Delta f \, f - f^2(|A|^2 + \bar R_{00}) + f^2 H^2 \, d\mu.$

In particular, if M^3_0 is a <u>minimal surface</u> ($H \equiv 0$), then M^3 is stable iff:

$$0 \le \delta^2(|M^n|, f) \quad \forall f \text{ with spt } f \text{ cpt}.$$

$$\Leftrightarrow \int_{M^n_0} (|A|^2 + \bar{Ric}_{00}) \le \int_{M^n_0} |\nabla f|^2 \, d\mu.$$

and b), if M^n_0 is a critical pt. of area w.r.t. same enclosed volume (w.r.t. to any fixed ref. surface)

i.e., M^n_0 is a critical pt of $\int_{M^n_t} d\mu + \lambda \, \mathrm{vol}(M^n_t) =: I(t)$, $\quad t \in (-\varepsilon, \varepsilon)$

$$\Rightarrow \delta(I(t); f)\big|_{t=0} = \int_{M^n_0} Hf \, d\mu + \lambda \int_{M^n_0} f \, d\mu = 0 \quad \forall f.$$

($H \equiv -\lambda$ CMC surface).

<u>Stable CMC</u>: $0 \le \delta^2(\Sigma(t), f)\big|_{t=0}$ $\forall f$ keeping volume constant. i.e. $\int_{M_0^n} f\,d\mu = 0$.

$$\Rightarrow 0 \le \int_{M_0^n} -\Delta f \cdot f - f^2(|A|^2 + \bar{R}_{00}) + H^2 f^2 \, d\mu + 2\int_{M_0^n} f^2 H \, d\mu$$

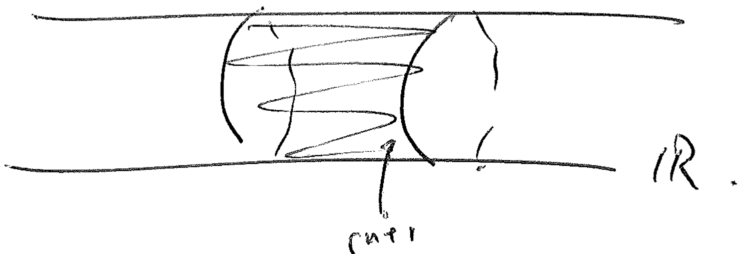
$$\Rightarrow \int_{M_0^n} f^2(|A|^2 + \bar{R}_{00}) \, d\mu \le \int_{M_0^n} |\nabla f|^2 \, d\mu$$ $\forall f$ with spt f cpct and $\int_{M^n} f \, d\mu = 0$.

<u>Rmk</u>. The formulas (I) & (II) are valid for just a <u>given</u> cptly spted speed f.

So, <u>stability</u> is a quantification over all such cptly spted f.

$$L_{stab} f = -\Delta f - f(|A|^2 + \bar{R}_{00}).$$

<u>Examples</u>: (I). Take $S^{n-1} \subset \underbrace{S^n \times \mathbb{R}}_{N^{n+1}}$ $(\subset \mathbb{R}^{n+1})$.



\# "totally geodesic". $|A|^2 = 0$, $\bar{R}_{00} = 0$.

$L_{stab} f = -\Delta$, kernel $= \{\text{const}\}$.

$\Rightarrow$ weakly stable as a min. surface.

$\Rightarrow$ strictly stable w.r.t. const. mean curvature surface.

(II) $S^n \subset S^{n+1}$ $(\subset \mathbb{R}^{n+2})$

$|A|^2 \equiv 0$, $R_{00} = n$.

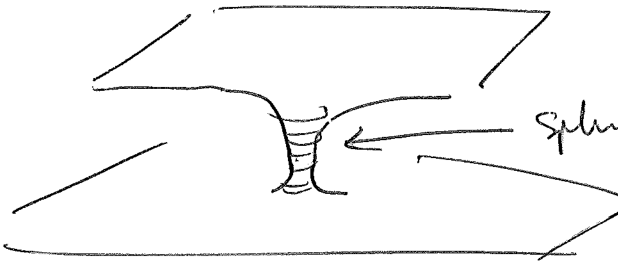
$L_{stab} = -\Delta f - n f$.

~~st~~ unstable as a minimal surface;

- ~~stable as~~ weakly stable as a CMC surface.

$$\boxed{\lambda_0(-\Delta) = 0 \quad , \quad \lambda_1(-\Delta) = n}$$

(III) $g = \delta \left(1 + \frac{m}{2r}\right)^4$.



spheres $\{r = r_0\}$ have CMC.

<u>Exercise</u>: (a) compute that $H = \dfrac{2}{r} \dfrac{\left(1 - \frac{m}{2r}\right)}{\left(1 + \frac{m}{2r}\right)^3}$

(b) $R_{\alpha\beta} = \dfrac{m}{r^3} \left(\delta_{\alpha\beta} - 3 \dfrac{y_\alpha y_\beta}{r^2} \right)$, $R = 0$.

Compute 1st eigenvalue of L_{stab} on f with

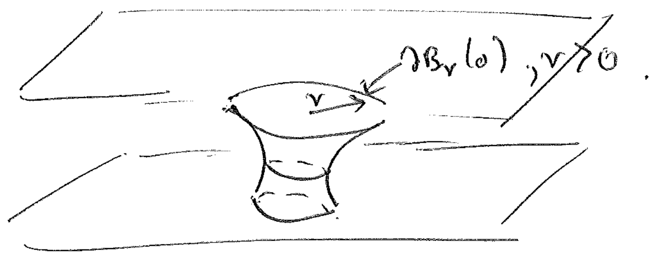
$\int f \, d\mu = 0$. $\quad 0 < \lambda_1 \approx \dfrac{6m}{r^5}$ as $r \to \infty$.

$\Rightarrow$ strictly stable if $m > 0$ as CMC surfaces.

<u>Rmk</u>. Stability for minimal surface $\to$ don't care about preserving volume. I.e., no need to consider only $\int f \, d\mu = 0$. But for CMC, this is a <u>must</u>.

CMC surfaces / foliations. — G. Huisken. 15/09/2015.

$$(N^3, g_m) = \{\mathbb{R}^3 \setminus \{0\},\ g_m = \delta\left(1 + \frac{m}{2r}\right)^4\}.$$



$$H(\partial B_r(0)) = \frac{2\left(1 - \frac{m}{2r}\right)}{r\left(1 + \frac{m}{2r}\right)^3}, \quad \text{compute} \quad |A|^2 = \frac{2\left(1 - \frac{m}{2r}\right)^2}{r^2\left(1 + \frac{m}{2r}\right)^6}.$$

$$\bar{R}_{00} = \frac{-2m}{r^3} \frac{1}{\left(1 + \frac{m}{2r}\right)^6}$$

$$L_{stab} f = -\Delta_g f - f\left(|A|^2 + \bar{R}_{00}\right).$$

$$\Rightarrow \text{on } \{f : \int f \, d\mu = 0\}, \quad \lambda_1(L_{stab}) = \frac{6m}{r^3\left(1 + \frac{m}{2r}\right)^6} = \frac{48\pi^{\frac{3}{2}} m}{|\partial B_r(0)|^{\frac{3}{2}}} > 0$$

<u>Note that:</u>
$$m = \frac{|\partial B_r(0)|^{\frac{1}{2}}}{(16\pi)^{3/2}} \left(16\pi - \int_{\partial B_r(0)} H^2 \, d\mu\right) \quad \forall r > 0.$$

<u>Hawking:</u> Given any $\Sigma^2 \subset (M^3, g)$ $(\hookrightarrow (L^4, h))$.

define
$$m_{Haw}(\Sigma^2) := \frac{|\Sigma|^{\frac{1}{2}}}{(16\pi)^{3/2}} \left(16\pi - \int_{\Sigma^2} |\vec{H}|^2 \, d\mu\right)$$

<u>Motivation:</u> If $\Sigma^2 = \partial\Omega$, $\Omega \subset (M^3, g)$. $m_{Haw}(\Sigma^2) \Leftrightarrow$ mass/energy contained in Ω.

"quasi local" mass.

• perfect on concentric spheres in spacelike Schwarzschild.

• In asymptotically flat (M^3, g) ~~then~~

$$\lim_{r\to\infty} m_{Haw}(\partial B_r(0)) = m_{ADM}(M^3, g).$$

• $\Sigma^2 \subset \mathbb{R}^3$: $\int H^2 d\mu \geq 16\pi$. Perfect only on round spheres, bad on all other surfaces.

<u>Thm</u> (Christodoulou–Yau, 1986).

Suppose $(N^3, \bar{g})$, C^∞ Riem, $\bar{R}(\bar{g}) \geq 0$. Let $\Sigma^2 \subset (N^3, \bar{g})$ is a smooth, closed surface of const. mean curvature which is <u>stable</u> (key assumption). Then, $m_{Haw}(\Sigma^2) \geq 0$.

and $=$ if $\Sigma = \partial\Omega$, $\Omega \subset (\mathbb{R}^3, \delta)$ i.e., round sphere in $\mathbb{R}^3$.

<u>Pf</u> Stable : $\int_{\Sigma^2} f^2(|A|^2 + \bar{R}_{00}) \leq \int_{\Sigma^2} |\nabla f|^2 d\mu$. $\forall f$, $\int f = 0$

Uniformisation ~~theorem~~ in $\mathbb{R}^3$

$\exists \varphi : \Sigma^2 \to S^2$ conformal diffeo,

Choose for $i = 1, 2, 3$, $\tilde{f}_i = x_i \circ \varphi$, and let $\tilde{c}_i = \int_{\Sigma^2} \tilde{f}_i \, d\mu_g \in \mathbb{R}^3$.

$\vec{c} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \end{pmatrix} \in \mathbb{R}^3$. "centre of gravity" of f^2 w.r.t. g.

$\exists \psi_i$ Möbius transformation of S^2 s.t. $f_i := x_i \circ \psi_i \circ \varphi$;

new centre $\vec{c} = 0 \in \mathbb{R}^3$, i.e. $\int_{\Sigma^2} f_i \, d\mu_g = 0$;

and $\sum_{i=1}^3 f_i^2 = 1$.

Use f_i in stability inequality, sum over i:

$$\int_{\Sigma^2} |A|^2 + \bar{R}_{00}\, d\mu \leq \sum_{i=1}^3 \int_{\Sigma^2} |\nabla f_i|^2\, d\mu_g.$$

$$= \sum_{i=1}^3 \int_{S^2} |\nabla x_i|^2\, d\mu_g = \sum_{i=1}^3 \sum_{k=1}^2 \langle e_k, \tau_i \rangle^2 d\mu_g.$$

(e_1, e_2) basis of TS^2.

normal $\nu = e_0$.

$$\nabla_{e_k} x^i = \nabla_{e_k} \langle e_0, \tau_i \rangle, \quad \{\tau_i\} \text{ basis of } \mathbb{R}^3.$$

$$\Rightarrow \boxed{\int |A|^2 + \bar{R}_{00}\, d\mu \leq 8\pi}$$

<u>Now note:</u> Gauß equations: $G = \bar{\sigma}_{12} + \lambda_1 \lambda_2$. in frame $e_1, e_2, \nu = e_0$.

$$\bar{R}_{00} = \bar{\sigma}_{10} + \bar{\sigma}_{20} + \bar{\sigma}_{12} - \bar{\sigma}_{12} = \tfrac{1}{2}\bar{R} - (G - \lambda_1 \lambda_2).$$

$$= \tfrac{1}{2}\bar{R} - G + \tfrac{1}{2}(H^2 - |A|^2), \quad |A|^2 = |\mathring{A}|^2 + \tfrac{1}{2}H^2.$$

$$= \tfrac{1}{2}\bar{R} - G + \tfrac{1}{4}H^2 - \tfrac{1}{2}|\mathring{A}|^2 + |\mathring{A}|^2 + \tfrac{1}{2}H^2.$$

$$\int \tfrac{1}{2}\bar{R} - G + \tfrac{3}{4}H^2 + \tfrac{1}{2}|\mathring{A}|^2\, d\mu \leq 8\pi.$$

$$0 \leq \int_{\Sigma^2} \tfrac{1}{2}\bar{R} + \tfrac{1}{2}|\mathring{A}|^2\, d\mu \leq 8\pi + 4\pi - \tfrac{3}{4}\int H^2\, d\mu.$$

$$= \tfrac{3}{4}\left(16\pi - \int H^2\, d\mu\right). \quad \square$$

Existence of CMC:

Sps that (M^3, g) has an asymptotically flat end satisfying suitable decay conditions on the metric:

$$g_{ij} = \delta_{ij} + p_{ij}, \quad p_{ij} = O(?), \ \nabla^l p_{ij} = O(?) \quad l \le 2 \ (?).$$

Assume $m_{ADM} = \frac{1}{16\pi} \lim_{R \to \infty} \int_{\partial B_R} (g_{ij,j} - g_{jj,i}) \nu^i \, d\sigma$

is well defined and positive (>0).

$\exists R_0 > 0$, and $\{\Sigma^2_\sigma\}_{0<\sigma<\sigma_0}$ $H(\Sigma^2_\sigma) = \sigma$, Σ^2_σ strictly stable

$M^3 \setminus B_{R_0}(P_0) \subset \bigcup \Sigma^2_\sigma$; Σ^2_σ form smooth foliation.

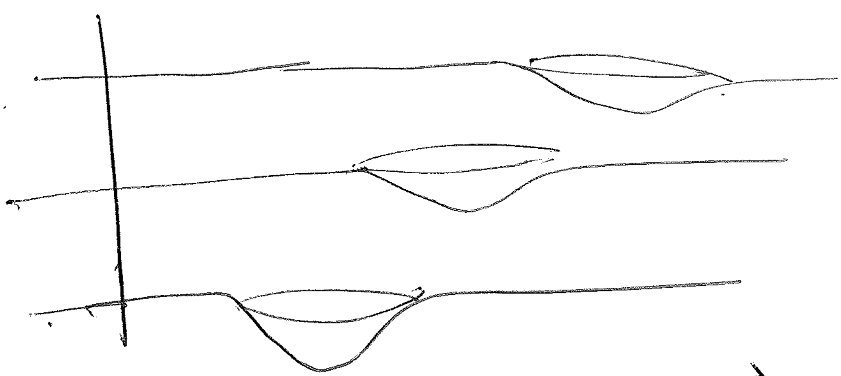
+ uniqueness statements also (need a little more).

Names: H.-S.T. Yau 1996, Metzger, Qin-Tian, Huang, Eichmair-Metzger, Bray, Bray-Morgan $\mathcal{Q}_{m_{Haw}}$

monotonicity of $m_{Haw}(\Sigma^2_\sigma)$ — isoperimetric property.

Cederbaum-Nerz, Nerz ~ 2014 $\rightsquigarrow$ centre of mass defined by $\{\Sigma^2_\sigma\}$.

$\downarrow$



Speed (asymptotically) of $\{\Sigma^2_\sigma\}$ should be $\vec{P}_m$.

True: C. Nerz.

Here $\vec{P}_v = \frac{1}{8\pi} \lim \int_{\partial B_r} k_{ij} \nu^j \, d\mu$.

Comments on existence:

(I) Can use implicit funct. H^n + strict stability

or can use a flow $\partial_t F = -(H-h)$, $h = ⨍ H \, d\mu$.

Studied for large coordinate spheres. M^3_+.

(II) Recall: ~~stability~~.

Stable $\Rightarrow$ $\int_{\Sigma^2} (\lambda_1 - \lambda_2)^2 = \frac{1}{2} \int |\mathring{A}|^2 \, d\mu$.

$$\leq \frac{3}{2} \left(16\pi - \int H^2 \right) \leq -\frac{m_{Haw}(\Sigma^2)}{|\Sigma|^{1/2}}$$

$$\leq c \cdot \frac{m_{ADM}}{r} \to 0.$$

$\rightsquigarrow$ bootstrap to higher order.

$\rightsquigarrow$ use spectral decomposition.

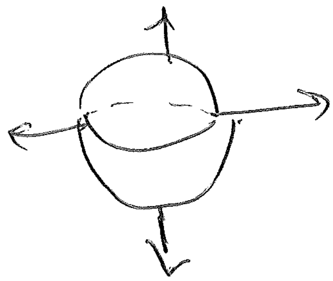
Rmk These are only in the exterior region.
But we would like this to go all the way.
(Problematic if two stars etc., then radius has to be very large.).

Inverse Mean Curvature flows.

Let $(N^{n+1}, \bar{g})$ be a smooth complete Riem mfld
Given $F_0 : M^n \hookrightarrow (N^{n+1}, \bar{g})$, $M_0^n = F_0(M^n) = \partial \Omega_0$,
$\Omega_0 \subset N^{n+1}$, with $H(M_0^n) > 0$, want to solve

(IMCF) $\partial_t F(p,t) = \frac{1}{H} \cdot \nu(p,t)$ (ν outer normal)

Example: $M^n_0 = S^n_{R_0} \subset (\mathbb{R}^{n+1}, \delta)$.



$\partial_t R(t) = \frac{1}{n} R(t)$, , $R(t) = R_0 \exp\left(\frac{t}{n}\right)$.

Note that: $\partial_t F = \frac{1}{H} \cdot \nu = - \frac{\vec{H}}{|\vec{H}|^2}$.

$$= \frac{1}{H^2} g^{ij} \left\{ \frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial F}{\partial x^k} + \bar{\Gamma}_{\beta\gamma} \frac{\partial F^\beta}{\partial x^i} \frac{\partial F^\gamma}{\partial x^j} \right\}$$

Linearisation RHS: $\frac{1}{H^2} \Delta$. (weakly elliptic).

(Same term as Einstein).

Thm: there smooth short time existence for any smooth closed initial M^n_0 with $H > 0$.

Note: $\partial_t g_{ij} = 2 \frac{h_{ij}}{H}$, $\partial_t g^{ij} = -2 \frac{h^{ij}}{H}$, $\boxed{\partial_t (d\mu) = d\mu}$

$$\boxed{\partial_t (d\mu) = (d\mu)}$$ Defining property, expands area exponentially:

$$\Rightarrow \quad |M^n_t| = |M^n_0| \exp(t).$$

$$\partial_t h_{ij} = -\nabla_i \nabla_j \left(\frac{1}{H}\right) + \frac{1}{H} \left(h_{ik} h^k_j - \bar{R}_{0i0j}\right).$$

$$\partial_t H = -\Delta\left(\frac{1}{H}\right) - \frac{1}{H}\left(|A|^2 + \bar{R}_{00}\right).$$

$$= \frac{1}{H^2} \Delta H - \frac{2}{H^3} |\nabla H|^2 - \frac{1}{H} (|A|^2 + \bar{R}_{00}).$$

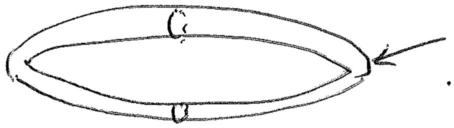
$$= D_i \left(\frac{1}{H^2} D_i H\right) - \frac{1}{H} (|A|^2 + \overline{Ric}_{00}).$$

<u>Note:</u> $|A|^2 \geq \frac{1}{n} H^2$ if $|\overline{Ric}| \leq C_0$.

$$\partial_t H \leq D_i \left(\frac{1}{H^2} D_i H\right) - \frac{1}{n} H + \frac{C_0}{H}.$$

So, <u>Prop</u>. $\sup_{M^n_t} H \leq \max\left(\sup_{M^n_0} H, \sqrt{\frac{C_0}{n}}\right)$.

<u>Ex</u>.



long thin torus, as $t \to \infty$. Something goes wrong.

The $-\frac{1}{H}(|A|^2 + \overline{Ric}_{00})$, term

makes $H \to 0$. So singularities.

But!!!

<u>Thm</u>. (Smoczyk ($n=2$), H. Ilmanen ($n \geq 3$))

As long as $H \geq \delta > 0$, the flow can be smoothly extended.

<u>Pf</u>. Max principle for $\Pi_{ij} = H h_{ij}$.

<u>Thm</u>. (Gerhardt) If $M^n_0 \subset (\mathbb{R}^{n+1}, \delta)$, $H > 0$, $\underbrace{\langle F, \nu \rangle > 0}_{\text{star shaped}}$,

then M^n_t expands to ∞, becoming round.

Main motivation for this flow:

Th$^{\underline{m}}$. (Geroch monotonicity of Hawking mass).

Sps $F: M^2 \times [0,T) \to (N^3, \bar{g})$ is sol$^{\underline{n}}$ of IMCF, and $\bar{R}(\bar{g}) \geq 0$. Then, $\partial_t m_{Haw}(M^2_t) \geq 0$, with "$=$" if

$M^2_0 \subset (N^3_{Schwarz}, g_m)$, M^2_t flows through concentric spheres.

Indeed: $\partial_t m_{Haw}(M^2_t) = \dfrac{|M^2_t|^{\frac{1}{2}}}{(16\pi)^{3/2}} \displaystyle\int_{M^2_t} 2|\nabla \log H|^2 + |\mathring{A}|^2 + \bar{R}\, d\mu$.

If $=0$, then can $|\nabla \log H|^2 = |\mathring{A}|^2 = \bar{R} = 0$. ← assign flat.

says flows through concentric circles. ← $\lambda_1 = \lambda_2 \Rightarrow$ umbilic.

Rmk this is outside the region, ie inside, anything possible.

Pf. Resembles Christodoulou–Yau: compute using $\partial_t H$, $\partial_t(d\mu)$:

$$\partial_t m_{Haw}(M^2_t) = \partial_t \left[\frac{|M_t|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \left(16\pi - \int H^2 d\mu\right) \right]$$

$$= \frac{1}{2} m_{Haw}(M^2_t) - \frac{|M_t|^{\frac{1}{2}}}{(16\pi)^{3/2}} \cdot$$

$$\int 2H \left(\nabla_i \left(\frac{1}{H^2} \nabla_i H \right) - \frac{1}{H} \left(|A|^2 + \bar{R}_{00} \right) + H^2 d\mu \right)$$

$$= \frac{1}{2} m_{Haw}(M^2_t) + \frac{|M_t|^{\frac{1}{2}}}{(16\pi)^{3/2}} \int 2|\nabla \log H|^2 + 2|A|^2 - H^2 - \bar{R}_{00}\, d\mu$$

$=$ as claimed! ← via Gauss eq$^{\underline{n}}$ & Gauss-Bonnet.

8

Ie, this is behaving as nicely as a CMC situation / foliation!.

Rmk. Exploits $\bar{R}(\bar{g}) \geq 0$, characterizes Schwarzschild, and relates small m_{Haw} to large m_{Haw} → which for nice mflds we know is close to m_{ADM}.

⇝ Hope to prove Penrose-inequality:

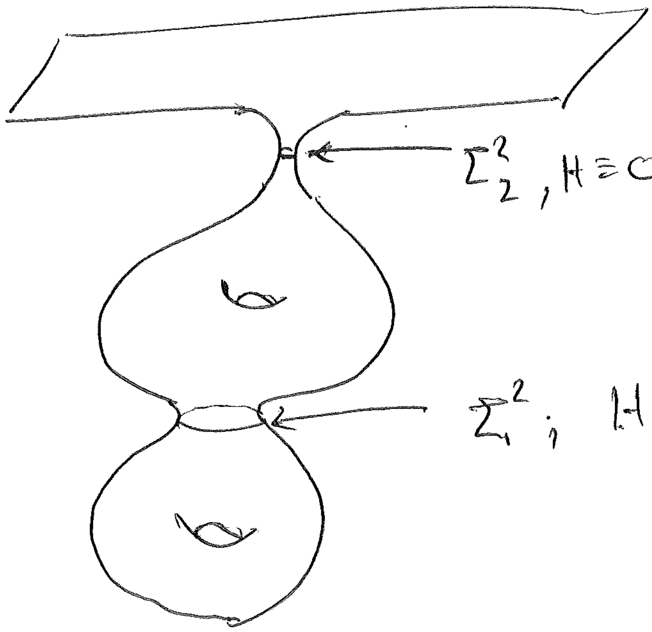
$$16\pi m^2_{ADM} \geq \underbrace{16\pi m^2_{Haw}(M^2_t)}_{t \text{ large}} \geq 16\pi m^2_{Haw}(M^2_{Horizon}) = |M^2_{Horizon}|.$$

"Mass on the exterior is larger than the mass in the black hole."

and ⇔ Schwarzschild.

This does not work: Need weak version of IMCF s.t. monotonicity is still true.

Problem.



$\Sigma^2_2, H \equiv 0.$ $16 m^2_{ADM} \geq |\Sigma^2_2|$

But

$\Sigma^2_1; H \equiv 0$ $16\pi m^2_{ADM} \ngeq |\Sigma^2_1|$

There cannot be a weak solution in this setting where $m_{Haw}(M_t^2)$ is monotone from Σ_1^2 all the way to ∂B_r, $r \gg 1$.

$\boxed{\text{!Need to allow jumps!}}$ $\longrightarrow$ needs to know to jump to better place to start.

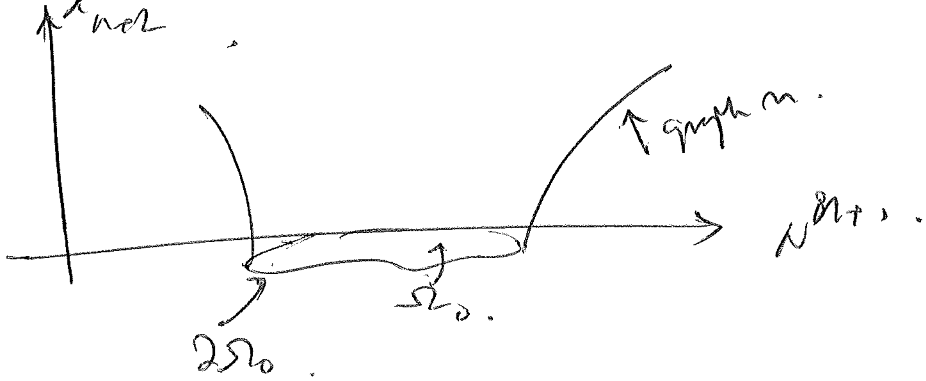
This cannot happen via a parabolic flow.

So, use a levelset formulation of IMCF:

Sps $M_t^2 = \{x \in N^3 : u(x) = t\}$, $u : N^3 \setminus \Omega_0 \to \mathbb{R}$.

$$\left(M_0^2 = \partial\Omega_0 = \{x \in N^3 ; u(x) = 0\}\right).$$

"$u(x)$ is the time when M_t^2 passes through $x \in N^3$."



If $|\nabla u(x)| \neq 0$: M_t^n is regular, $\nu = \dfrac{Du}{|Du|}$,

$$H = \mathrm{div}(\nu) = D_i\left(\frac{D_i u}{|Du|}\right).$$

Speed of levelset: $\dfrac{1}{|\nabla u|}$ $\searrow$ inverse of speed.

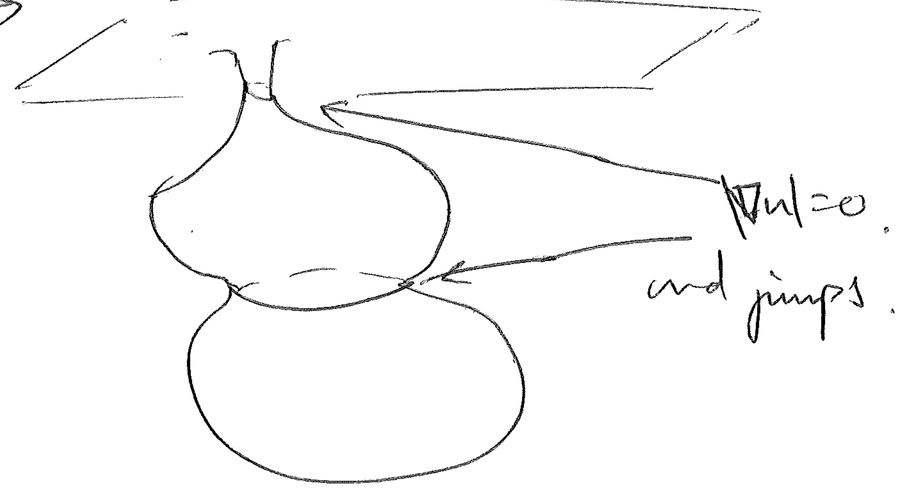
$$(\text{IMCF}) \iff \begin{cases} D_i\left(\dfrac{D_i u}{|Du|}\right) = |Du|. \\ u = 0 \quad \text{on } \partial\Omega_0. \end{cases}$$

$$D_i\left(\frac{D_i u}{|Du|}\right) = \frac{1}{|Du|}\left(\delta_{ij} - \frac{D_i u\, D_j u}{|Du|^2}\right) D_i D_j u.$$

$\to$ Degenerate elliptic equation $\to$ degeneracy in the direction of the normal.

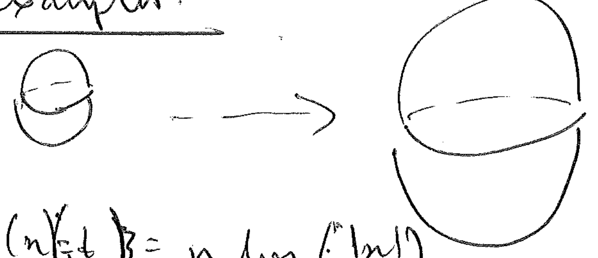
Need to allow $|\nabla u(x)| = 0$. to solve the problem.

Ie, allow plateau's.



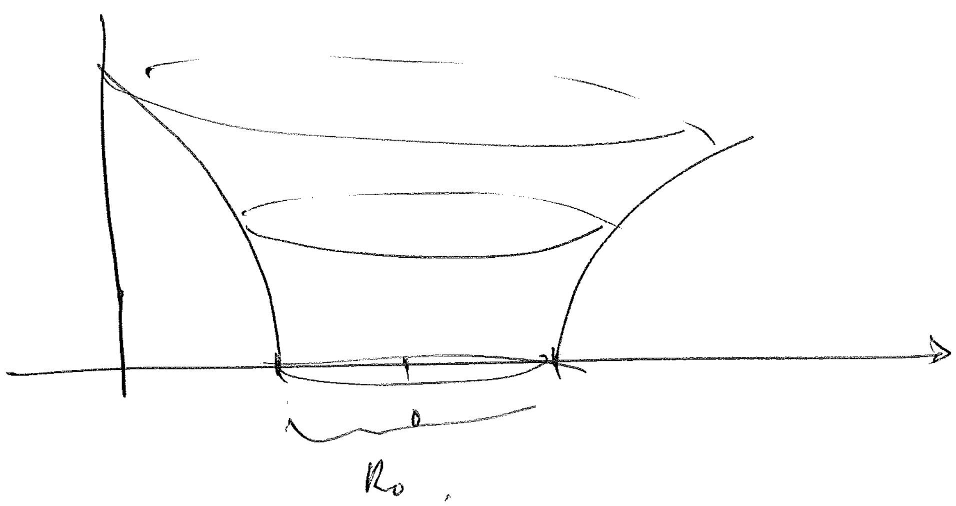
Main examples:

(I)

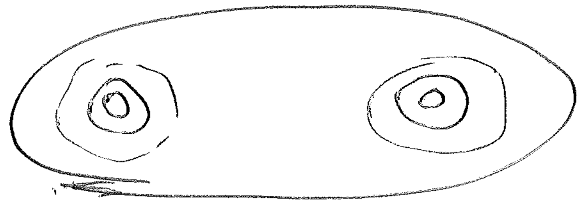


(II) $u(x,t) = u \log\left(\frac{|x|}{R_0}\right)$.

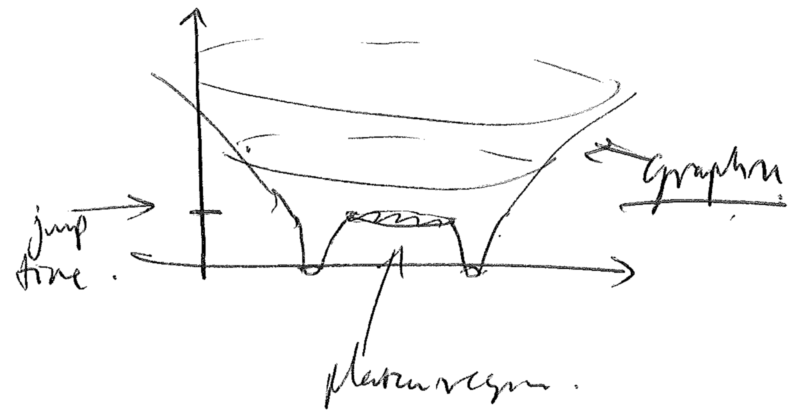
(III)



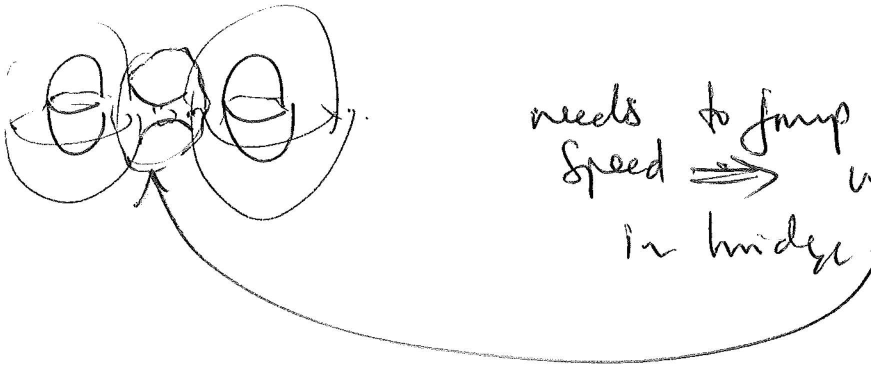
(II).



jumps.



(III).



needs to jump at infinite speed $\Longrightarrow$ need to glue in bridge (catenoid).

To get this behavior, u needs to satisfy a variational principle: want (weakly)

$$D_i\left(\frac{D_i u}{|Du|}\right) = |Du|.$$

Variation means this should be Euler-Lagrange of a functional.

Define.

$$F^K_u(v) := \int_K \underbrace{|Dv|}_{\text{LHS}} + \underbrace{v\cdot|Du|}_{\text{Bounded, to get RHS (no way of doing that variationally)}}. \quad K \subset\subset N^{n+1}.$$

Want: $u \in C^{0,1}(N^{n+1})$ s.t.

$$F^K_u(u) \le F^K_u(v). \quad \forall v \text{ with } \operatorname{spt}(v-u) \subset K.$$

"relative variational principle." or "stability prop of u."

If u is a sol^n of this variational principle.

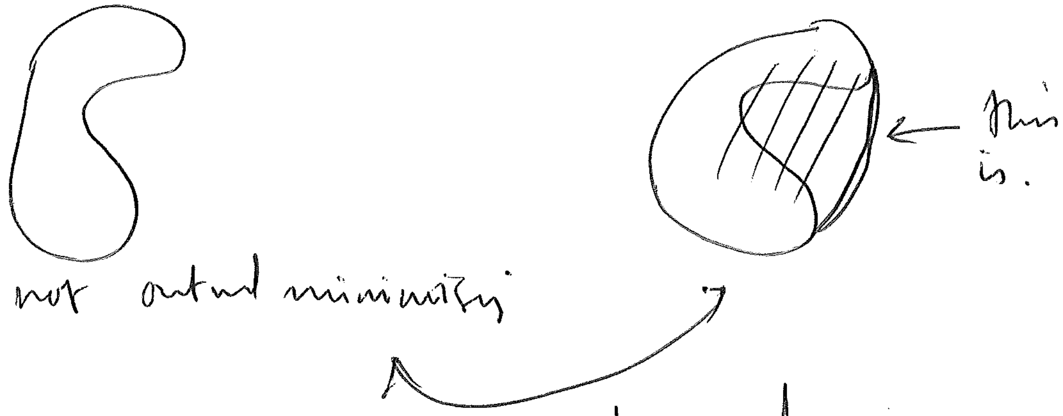
$\Rightarrow$ the surface $M_t^2 = \partial\{x \in N^{n+1} : u(x) < t\}$.

is <u>outward minimising</u>.

i.e. If $M_t^2 = \partial \Omega_t$ and $\tilde{\Omega}$ is a region $\tilde{\Omega} \supset \Omega_t$, $\Sigma^n = \partial\tilde{\Omega}$. then.

$$|\Sigma^n \cap E| \geq |M_t^n \cap E|.$$

For curves:



explains how and why jump happens.

M_t^2 is then continuous in t $\forall t > 0$.

$|M_t^2| = |M_0^n| \exp(t)$ if M_0^n is itself outward minimising.

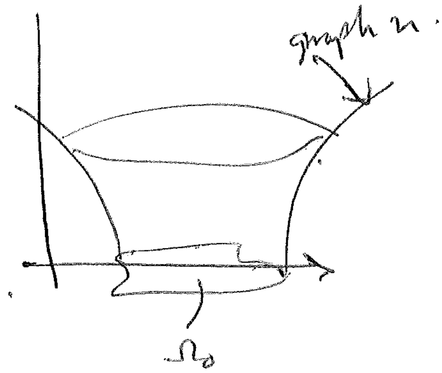
In Hawking mass $-\int H^2 d\mu$ term gets replaced and Hawking mass jumps up. Ie, due to this variational principle!

Inverse Mean Curvature Flow. G. Huisken. 16/09/2015.

$(N^3, \bar{g})$, smooth Riem 3-mfld, asymp flat.

$M_0^2 = \partial\Omega_0 \quad \Omega_0 \subset (N^3, \bar g)$. Want to solve:

(IMCF) $\partial_t F = \frac{1}{H}\nu$



$\overset{\text{weak}}{\Longleftrightarrow}$ $\begin{cases} D_i\left(\dfrac{D_i u}{|Du|}\right) = |Du| & \text{in } (N^3,\bar g)\setminus\Omega_0 \\ u = 0 & \text{on } \partial\Omega_0 \end{cases}$

for $\tilde F_u^k(v) = \int_K |Dv| + v|Du| \, dvol$, $\tilde F_u^k(u) \le \tilde F_u^k(v)$

$\forall v$ s.t. (~~xxxx~~) cts.

Elliptic regularisation. (w/ T. Ilmanen).

(Later, R. Moser: p-harmonic).

(ε) $\begin{cases} D_i\left(\dfrac{D_i u_\varepsilon}{\sqrt{\varepsilon^2+|Du_\varepsilon|^2}}\right) = \sqrt{\varepsilon^2 + |Du_\varepsilon|^2} & \text{in } B_{\frac{1}{\varepsilon}}(p_0)\setminus\Omega_0 \\ u_\varepsilon = 0 & \text{on } \partial\Omega_0 \\ u_\varepsilon \text{ "=" } \log\left(\frac{1}{\varepsilon}\right) & \partial B_{\frac{1}{\varepsilon}}(0) \end{cases}$

$u_\varepsilon \in C^{2,\alpha}(B_{\frac{1}{\varepsilon}}\setminus\Omega_0)$, $u_\varepsilon \ge -\varepsilon$.

(Need to construct a barrier, because sol$^{\text{n}}$ is barely positive,
$\rightarrow$ use asymptotic flatness).

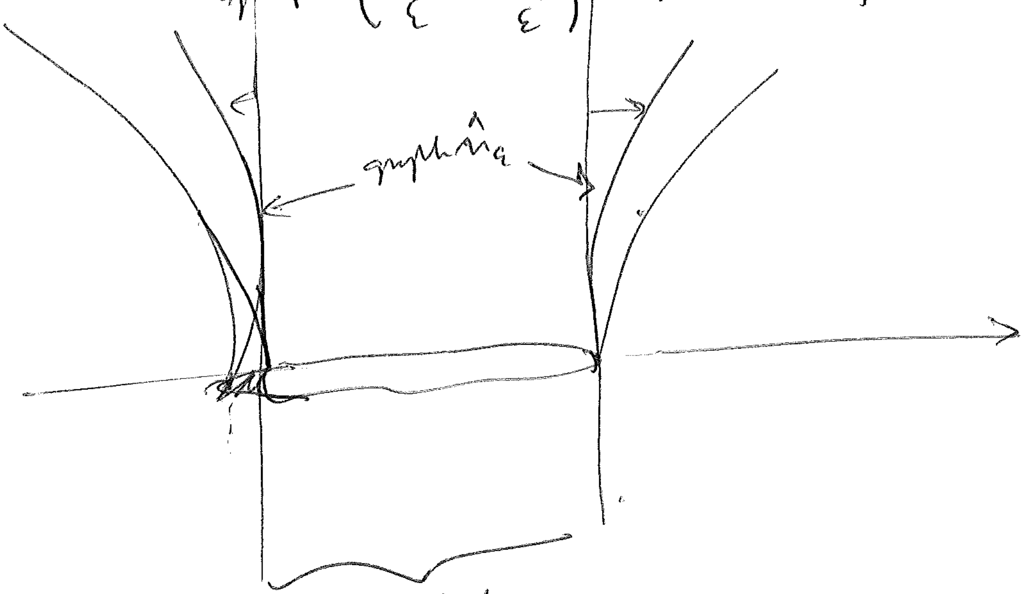
Set $\hat{u}_\varepsilon := \frac{u_\varepsilon}{\varepsilon} \Rightarrow$.

$(\hat{\varepsilon})$. $$D_i\left(\frac{D_i \hat{u}_\varepsilon}{\sqrt{1+|D\hat{u}_\varepsilon|^2}}\right) = \varepsilon\sqrt{1+|D\hat{u}_\varepsilon|^2}.$$

Geom int.

$\Leftrightarrow$ $H(\text{graph}\,\hat{u}_\varepsilon) = \varepsilon\langle \hat{\nu}_\varepsilon, \frac{\partial}{\partial z}\rangle^{-1}$.

$\Leftrightarrow$ graph $\left(\frac{\hat{u}}{\varepsilon} - \frac{t}{\varepsilon}\right)$ moves by IMCF!!!!] Elliptic reg. is not just a technical trick.



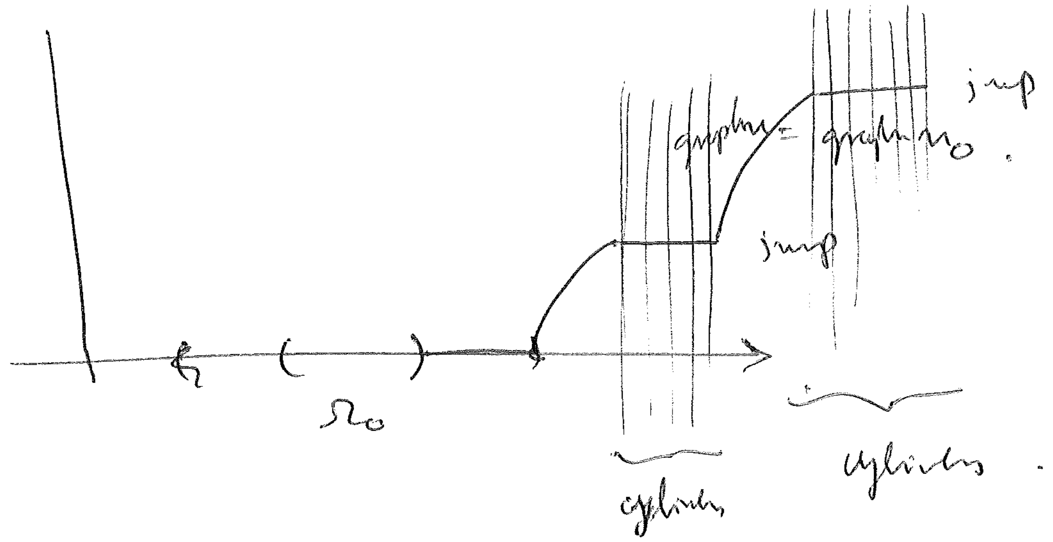
Claim: cylinder in limit; (still sol^n to IMCF in ambient mfld), ~~Re~~

Get estimates indep of $\varepsilon > 0$: $\cdot\ |\hat{H}_\varepsilon| \le C_0$.

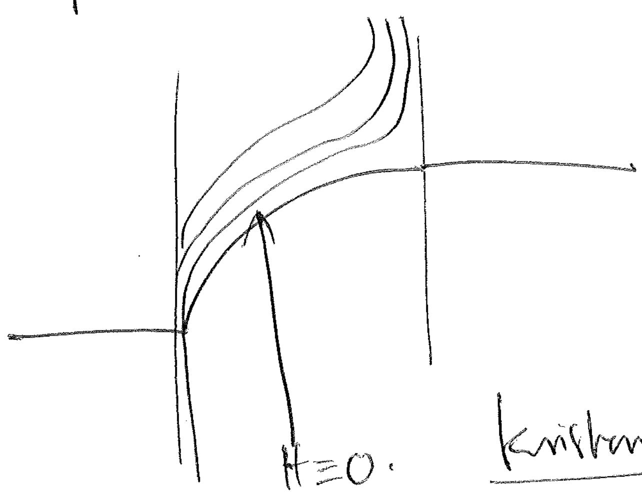
$\cdot\ \int_{t_1}^{t_2}\int_{\text{graph}\,\hat{u}_\varepsilon} |\nabla \log H|^2 + |A|^2 \,d\mu\,dt \le C_1$.

$\rightsquigarrow$ take limit to conv to vertical cylinders wherever $\hat{H}_\infty > 0$. ($\Leftrightarrow |Du_\infty| > 0$).

+ jump regions.



$\Rightarrow$ prove Geroch monotonicity in $\varepsilon \to 0$ limit.



Kristen Moore 2012

$\Rightarrow$ more structure classifications, well defined unit normal $\Rightarrow$ Project $\Rightarrow$ well defined vector field.

$\Rightarrow$ Extend IMCF to null-IMCF.

$$\partial_t F = \frac{1}{H \pm p}\,\nu$$

where $p = \mathrm{tr}_2 \{k_{ij}\}$ (the trace of 2nd ff).

on $(N^3, \bar{g}, k = \{k_{ij}\})$ (useful for horizons).

$$D_i\left(\frac{D_i u}{|Du|}\right) + \underbrace{k_{ij}\frac{D_i u}{|Du|} \cdot \frac{D_j u}{|Du|}}_{} = |Du|.$$

Moore: Existence of sol^{n}s. Soft. var. principle.

Moore makes sense of this since $u = 0$.

Open: Geroch monotonicity-analogue.

Corollary. Let $M_0^2 = \partial\Omega_0$, $\Omega_0 \subset (N^3,\bar g)$,
$R(\bar g)\geq 0$, $\partial\Omega_0$ outward minimising and connected. Then,

$$m_{Haw}(M_0^2) \leq m_{ADM}(N^3,\bar g).$$

In particular, if $(N^3,\bar g)\neq(\mathbb{R}^3,\delta)$ then $m_{ADM}>0$.

($\Rightarrow$ Another Pf. of positive mass thm).

Also, if Σ_0^2 is a component of the outermost horizon, then

$$\boxed{m_{ADM} \geq m_{Haw}(\Sigma_0^2) \Leftrightarrow 16\pi m_{ADM}^2 \geq |\Sigma_0^2|}$$

Penrose Ineq.

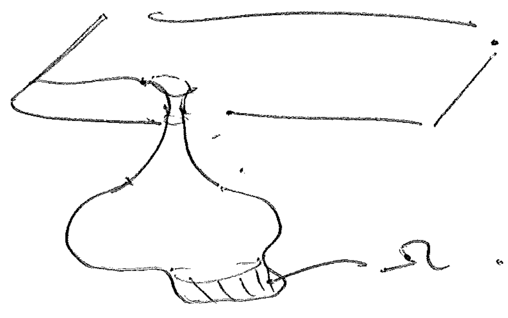
H; Bray's: replace $|\Sigma_0^2|$ by $\sum_k |\Sigma_k^2|$, but no foliation.

Application to Bartnik capacity

Def^n: Let (Ω,g) be a 3-Riem mfld, cpct with boundary, $R(g)\geq 0$. Then,

$$\mathrm{cap}_B(\Omega,g) := \inf\Big\{ m_{ADM}(N^3,\bar g) : (\Omega,g) \xhookrightarrow{\text{isometric}} (N^3,\bar g), \; R(\bar g)\geq 0, \text{ no minimal surfaces in } N^3\setminus\Omega \Big\}$$

Rules out



(I) $(\tilde\Omega,\tilde g) \supset (\Omega, g)$ $\Rightarrow$ $\mathrm{cap}_B(\tilde\Omega,\tilde g) \geq \mathrm{cap}_B(\Omega,g)$.

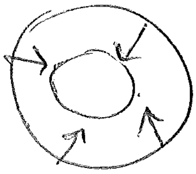
(II) $\mathrm{cap}_B(\Omega,g) \geq 0$ + $\underbrace{\mathrm{cap}(\Omega,g)>0 \text{ unless } (\Omega,g)\hookrightarrow(\mathbb{R}^3,\delta)}_{\text{By Thm: long time open problem.}}$

<u>Mean Curvature Flow in 3-mflds</u>

Given $M_0^2 \subset (N^3,\bar g)$, $F: M_0^2 \longrightarrow (N^3,\bar g)$, solve

$$\partial_t F(p,t) = \vec H(p,t) := -H(p,t)\cdot\nu(p,t) \qquad \nu \text{ - outer normal.}$$
$$= \Delta_{g(t),\bar g} F(p,t).$$

<u>Examples:</u>



$\partial_t R(t) = -\frac{2}{R(t)}$, $R(t) = \sqrt{R_0^2-4t}$,

$T_{max} = R_0^2/4$.

- $\partial_t g_{ij} = -2H h_{ij}$, $\partial_t (d\mu) = -H^2 (d\mu)$

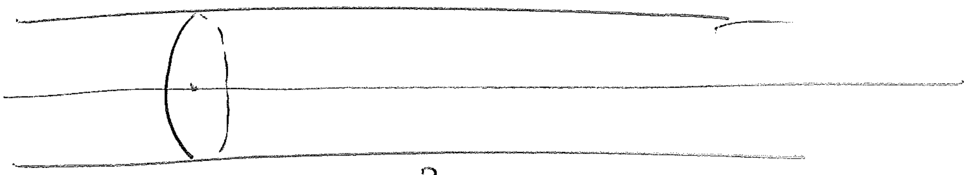
So; $\underbrace{\partial_t |M_t^2| = -\int H^2 \, d\mu}$.

In Euclidean space, spheres are a barrier, so $T_{max} = R/4$ when $B_R(0) \supset M_t^2$.

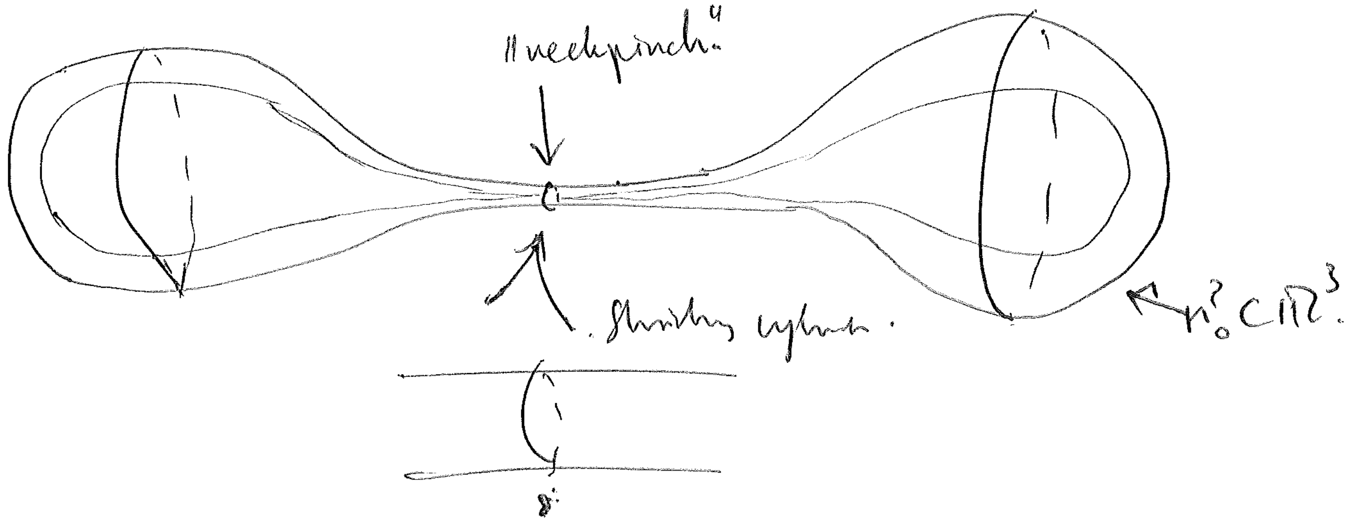
But, if it exists for all time in a general manifold, then, tends to minimal surface. because $\partial_t |M_t^2| = -\underbrace{\int H^2 \, d\mu}_{\to 0} \Rightarrow H_t \to 0$ ..

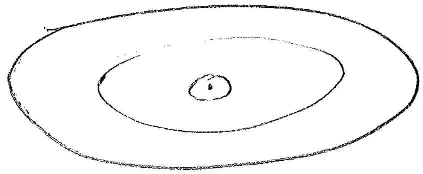
$$\partial_t H = \Delta H + H\left(|A|^2 + \overline{Ric}(\nu,\nu)\right).$$

Singularities:



$S^1 \times \mathbb{R} \subset \mathbb{R}^3$.



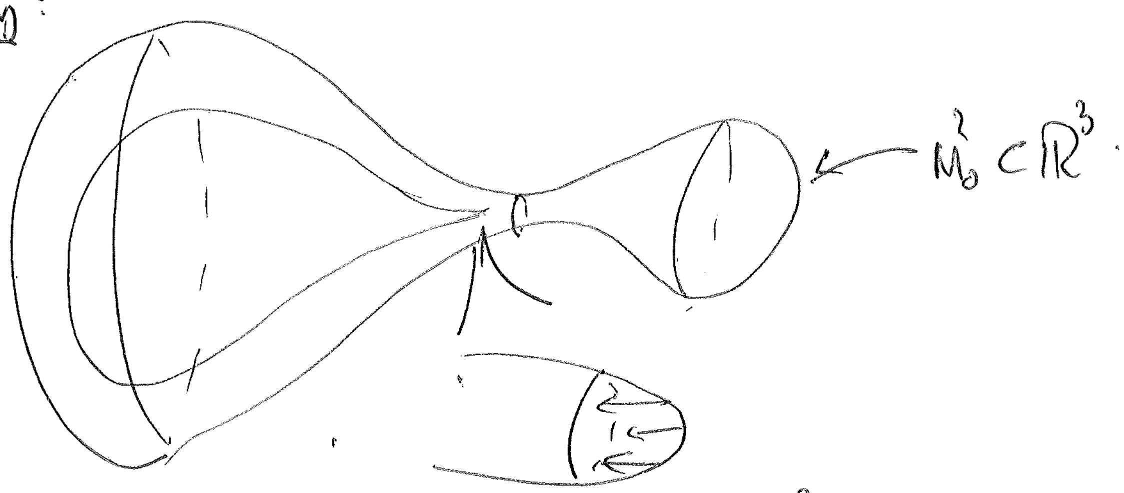


$M_0^2 \subset \mathbb{R}^3$. convex $\Rightarrow$ M_t^2 remain convex

and $M_t^2 \to \{p\}$.

So eventually becomes a shrinking sphere.

Nasty:



Too large for convex, too small for neckpinch.

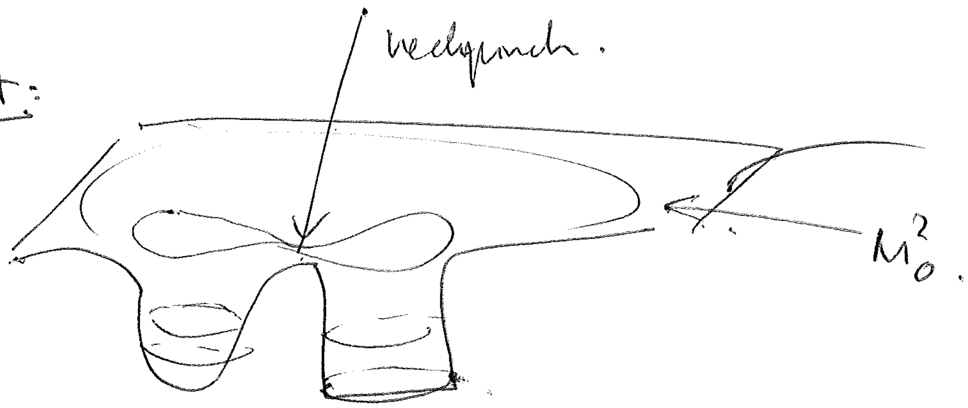
$\Rightarrow$ cusp. (translating solution).

that slightly away from the neck, you get the shrinking cylinder.

<u>Thm</u>. (S. Brendle, H. 2015).

Let $M_0^2 \subset (N^3, \bar{g})$ positive m.c. and <u>embedded</u>, then these are the only singularities, (with precise quantitative estimate → i.e. how big under: has to be with initial data until you see these).

<u>Want:</u>

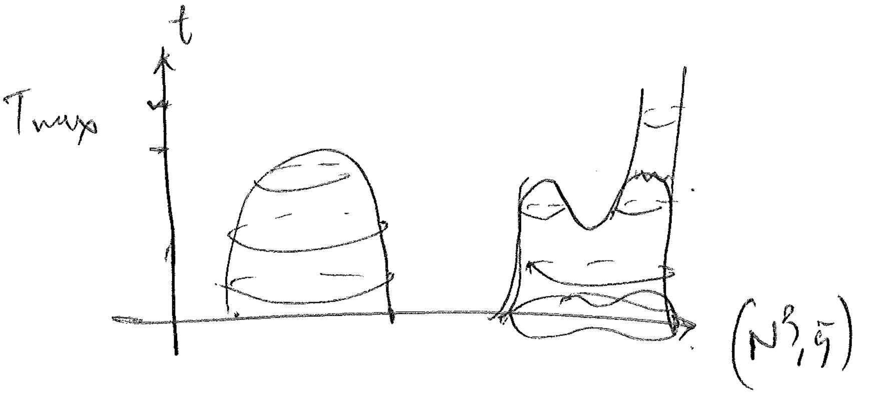


2 possibilities: . level set flow.
. "flow" with surgery.

<u>Level-set flow:</u> ($\underbrace{\text{Chen – Giga – Goto, Evans – Spruck}}_{}$). $u \in C^{0,1}$.

$$M_t^2 = \{ x \in (N^3, \bar{g}) : u(x) = t \}. \quad u\big|_{\partial\Omega_0} = 0.$$

$$D_i\left(\frac{D_i u}{|Du|}\right) = -\frac{1}{|Du|} \quad \text{in } \Omega_0, \; M_0^2 = \partial\Omega_0.$$



Reg. theory for level set flow, B. White.

$n=2$: Sol^{n} C^∞ for t-a.e. $t>0$.

<u>Elliptic regularisation</u>. M_0 outward minimising.

$$D_i\left(\frac{D_i u_\varepsilon}{\sqrt{\varepsilon^2+|Du_\varepsilon|^2}}\right) = \frac{-1}{\sqrt{\varepsilon^2+|Du_\varepsilon|^2}}, \quad \hat u_\varepsilon = \frac{u_\varepsilon}{\varepsilon}.$$

$$\Rightarrow D_i\left(\frac{D_i \hat u_\varepsilon}{\sqrt{1+|Du_\varepsilon|^2}}\right) = -\frac{1}{\varepsilon}\langle \hat\nu_\varepsilon, \frac{\partial}{\partial z}\rangle. \quad \text{MCF in } N^3\times\mathbb{R}.$$

This is where the analogy stops with IMCF.

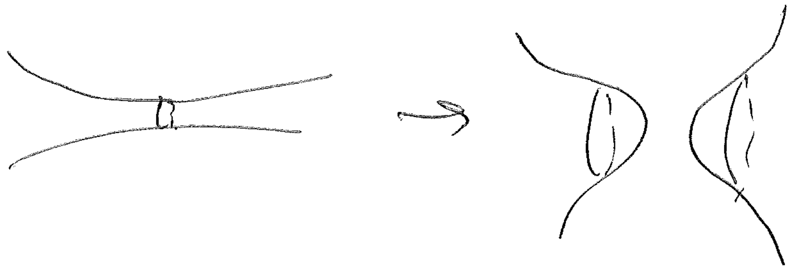
<u>Flow with surgery:</u>

Given $M_0^2 \subset (N^3, \bar g)$, embedded with $H>0$, ~~$\exists\, 0<t_1<t_2<\ldots<T_{max}$~~

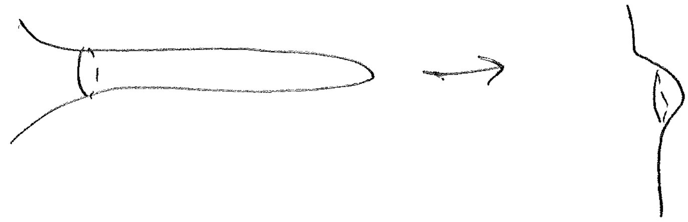
$\exists\; 0 < t_1 < t_2 < \ldots < t_N < T_{max} \leq \infty$.

(finitely many surgery times). and $F_i : M_i^2 \times [t_{i-1}, t_i] \to (N^3, \bar g)$.

Satisfying C^∞ m.c.f., s.t. $M_i^2(t_{i-1})$ is obtained from $M_{i-1}^2(t_{i-1})$ by either:



or.



or, removing tiny spheres, rings.

As $t \to t_N$, M_t^2 is completely smooth.

If $T_{max} = \infty$, M_t^2 converges to a weakly stable minimal surface.

So, it means that, after finitely many accidents, the flow will find the outermost horizons + complete control of the topology of the 3-mfd along the flow:

<u>Corollary</u> If $(N^3, \bar{g})$ is asympt. flat, and ~~$M_0^2 = \partial B_R(0)$~~ $M_0^2 = \partial B_R(0) \subset (N^3, \bar{g})$, then M_t^2 converges after finitely many surgeries to the outermost horizon (min surface) in $(N^3, \bar{g})$. If $\bar{R}(\bar{g}) \geq 0$, this horizon is a disjoint collection of spheres. (use Stab. trick from before $\Rightarrow \int_{\Sigma^2_\infty}$ (G dμ)).

R - chosen large so that $\underset{\text{a priori embedded}}{\overset{M_0^2 \downarrow \text{embedding}}{\partial B_R(0)}} \xhookrightarrow{\cancel{\text{embedded}}} (N^3, \bar{g})$.

In progress: (; w/ C. Sinestrari):

Can let $R_i \to \infty$, ${}^iM_0^2 = \partial B_{R_i}(0)$.

Claim If $m_{ADM}(N^3, \bar{g}) > 0$ and $(N^3, \bar{g})$ appropriately asymp. flat, then ${}^iM_t^2$ converge to a MCF. Sol^n. ${}^\infty M_t^2$, defined for $t \in (-\infty, +\infty)$:

$M_t^2 \to$ spatial infinity, $t \to -\infty$.
$M_t^2 \to$ horizon, $t \to \infty$.

($\sim$ "Nonlinear Newton potential").

Rmk Uniqueness is for the level set flow:
Head / other find that if you let the scale at which you do the flow $\to 0$, then converge to ~~[illegible]~~ level-set flow.

Recall $m_{ADM}(N^3, \bar{g}) = \frac{1}{16\pi} \int_{S^2_\infty} (g_{ii,j} - g_{ji,i}) \nu^j ds$. &

~~$R \geq 0$~~ $\bar{R}(\bar{g}) \geq 0$.

Th^m (I) $m_{ADM}(N^3, \bar{g}) = m_{iso}(N^3, \bar{g}) = \limsup_{\substack{\Omega_i \nearrow N^3 \\ \text{outer min}}} \left(\frac{2}{|\partial\Omega_i|} \left(\mathrm{vol}\,\Omega_i - \frac{1}{6\pi^{\frac{1}{2}}} |\partial\Omega_i|^{\frac{3}{2}}\right) \right)$.

"iso perimetric - deficit"

(II) $m_{ADM}(N^3, \bar{g}) = \sup_{\substack{\Omega \subset (N^3, \bar{g}) \\ \text{outer min.}}} m_{Haw}(\partial\Omega)$.

$m_{Haw}(\Sigma^2) = \frac{|\Sigma|^{\frac{1}{2}}}{(16\pi)^{3/2}} \left(16\pi - \int H^2 d\mu\right)$. "willmore deficit."

Consider the isoperimetric profile of Schwarzschild.

$$g_m = \delta\left(1+\frac{m}{2r}\right)^4 \quad m>0.$$

$$\Phi : [16\pi m^2, \infty) \to [0,\infty).$$

$$\text{Area}(\partial B_r),\ r \geq m/2 \quad \text{vol}\left(B_r(0)\setminus B_{m/2}(0)\right).$$

(I) $$\Phi_m(s) = \underbrace{\frac{s^{3/2}}{6\pi^{1/2}}}_{\text{isoperimetric profile of } \mathbb{R}^3} + \frac{1}{2} m\cdot s + \text{l.o.}$$

(II) $$\Phi'_m(s) = \frac{s^{1/2}}{4\pi^{1/2}}\left(1-\frac{4\pi^{1/2} m}{s^{1/2}}\right)^{-1/2}.$$

Now, let M^2_t be sol^n of MCF in arbitrary $(N^3, \bar{g})$. $M^2_t = \partial\Omega_t$.

$$\partial_t\left(\Phi_{m_{ADM}}(|M^2_t|) - \text{vol}(\Omega_t)\right)$$

$$= \Phi'_{m_{ADM}}(s)\cdot\left(-\int_{M_t^2} H^2\, d\mu\right) + \int_{M_t^2} H\, d\mu.$$

Hölder $\Rightarrow \int H \leq \left(\int H^2\, d\mu\right)^{1/2} \cdot \underbrace{|M_t^2|^{1/2}}_{s^{1/2}}$

relate $\int H^2\, d\mu \to m_{Hawk} = \frac{|M_t^2|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int H^2\, d\mu\right).$

+ ODE

$\Rightarrow \partial_t(\text{———}) \leq 0$ provided $\boxed{m_{Hawk}(M^2_t) \leq m_{ADM}}$

↑ true IMCF.

We have $M_{Haw}(M^2_t) \geq m_{ADM}$ by applying IMCF to each sol^{n} of M^2_t of MCF.

By White M^2_t outer minimising $\Rightarrow$

$$\Phi(|M^2_{t_1}|) - \mathrm{vol}(\Omega_{t_1}) \leq \Phi_{m_{ADM}}(|M^2_0|) - \mathrm{vol}(\Omega_0).$$

$$= \frac{|M^2_0|^{3/2}}{6\pi^{\frac{1}{2}}} + \frac{1}{2} m_{ADM} |M^2_0| + o(|M^2_0|) - \mathrm{vol}(\Omega_0).$$

$$\Rightarrow \quad \frac{2}{|M^2_0|}\left(\mathrm{vol}\,\Omega_0 - \frac{|M_0|^{3/2}}{6\pi^{\frac{1}{2}}}\right)$$

$$\leq \frac{o(|M^2_t|)}{|M^2_0|} + \boxed{m_{ADM}} + .$$

$$\frac{\mathrm{vol}(\Omega_t)}{|M^2_0|} - \Phi_{m_{ADM}}(|M^2_{t_1}|)\frac{1}{|M^2_0|}.$$

letting $t_1 \to \infty$, get that isoperimetric deficit is bdded by m_{ADM}!

uses MCF + IMCF.