Conformal Scattering

1965 – Penrose.

1967: Lax–Phillips:

Plane wave decomposition of the field, (sol^n to free wave.)

fourier transform away from obstacle.

$$\varphi_0^+(s,\omega) = \lim_{r \to +\infty} \frac{1}{r} \partial_t u(r, (r+s)\omega).$$

$$u(t,x) = \frac{1}{2\pi} \int_{S^2} \varphi_0^+(x, \omega + t, \omega) \, d\omega.$$

1970's, 1980: Friedlander:

1980's 1990's: Baez et. al.

2002, 2010: Mason, N. Jardinusse.

Spectral Scattering

1980's: Dimock, Kay, Bachelot, N., Melnyk.
C. Gerard, D. Häfner, D. H, N, T. Daudé.

Hawking: Bachelot, Melnyk, Häfner.

N.B. Wave operators and scattering.

H, H_0 s.a. operators on Hilbert spaces $\mathcal{H}$, $\mathcal{H}_0$.

eqs.
$$\begin{cases} \partial_t \Psi = iH\Psi & \text{full} \\ \partial_t \varphi = iH_0 \varphi. & \text{simplified.} \end{cases}$$

$\forall \psi_0 \in \mathcal{H}, \ \exists!\ \varphi_0^\pm \in \mathcal{H}$ s.t. $\lim_{t\to\pm\infty} \| e^{itH}\psi_0 - J e^{itH_0}\varphi_0^\pm \|_{\mathcal{H}} = 0$

$J : \mathcal{H}_0 \to \mathcal{H}$.

$W^\pm : \varphi_0^\pm \longmapsto \psi_0$, $\tilde{W}^\pm : \psi_0 \longmapsto \varphi_0^\pm$.

$S = \tilde{W}^+ W^-$.

Example: Wave eq^n in an asymptotically flat spacetime.

Comparison dynamics or equations:

1) Wave eq. in M : $\mathcal{H}_0 = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.
2) dynamics: $\varphi_0^\pm(s-t,\omega)$, $\dot{H}^1(\mathbb{R}, L^2(S^2))$.

↖ not comparable norms, but still ok!

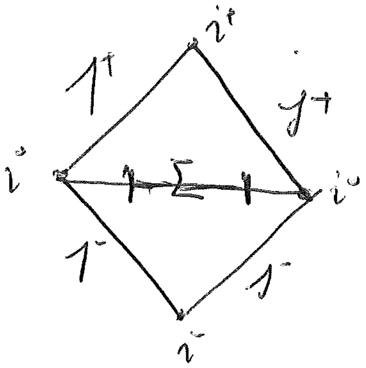
Conformal Scattering.

Conformal wave eq^n: $\Box_g u + \frac{1}{6}\,\mathrm{Scal}_g\, u = 0$.

(I) Conformally cpct spacetime
$(M = \mathbb{R}^4, g)$, smooth conformal compactification $(\bar{M}, \bar{g})$.
$\bar{g} = \Omega^2 g$, Ω - satisfies good properties.

$\bar{M} = M \cup \mathscr{I}^\pm \cup \{i^\pm\} \cup \{i^0\}$.

points - point



$u \in \mathcal{D}'(\mathbb{R}^4)$. $\Box_g u + \frac{1}{6}\,\mathrm{Scal}_g\, u = 0$.

$$\Updownarrow$$

$\tilde{u} = \Omega^{-1} u$. $\Box_{\tilde{g}} \tilde{u} + \frac{1}{6}\,\mathrm{Scal}_{\tilde{g}}\, \tilde{u} = 0$.

Σ Cauchy hypersurface for (M, g).

$\varphi_0^\pm = \tilde{u}/\mathscr{I}^\pm$ assuming data for u are in $C_0^\infty(\Sigma)$

<u>1st Step:</u> $T^+ : C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) \to C^\infty(\mathscr{I}^+)$.

linear $\Psi_0(\tilde{u}|_{t=0}, \partial_t \tilde{u}|_{t=0}) \mapsto \varphi_0^+ = \tilde{u}/\mathscr{I}^+$.

<u>2nd step:</u> energy estimates both ways b/w Σ and $\mathscr{I}^+$.

$T_{ab} = \nabla_a \tilde{u} \nabla_b \tilde{u} - \frac{1}{2} \langle \nabla \tilde{u}, \nabla \tilde{u} \rangle_{\tilde{g}}\, \tilde{g}_{ab}$

Choose τ^a timelike on M (causal on $\bar{M}$).

$J^a = \tau^a T_{ab}$, energy current!

1st choice: τ timelike on $\bar{M}$ & on $\mathscr{I}^+$; $\mathcal{H}_0 \simeq H^1(\mathscr{I}^+)$.

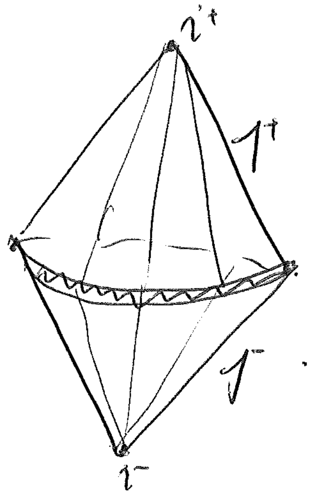
2nd choice: τ tangent to $\mathscr{I}^+$; on $\mathscr{I}^+$, $\mathcal{H}_0 = H^1(\mathbb{R}, L^2(S^2))$

$\|T^+ \Psi_0\|_{\mathcal{H}_0} \simeq \|\Psi_0\|_{\mathcal{H}} \Rightarrow T^+ \in \mathcal{L}(\mathcal{H}, \mathcal{H}_0)$. (from Stokes Th$^{=}$)

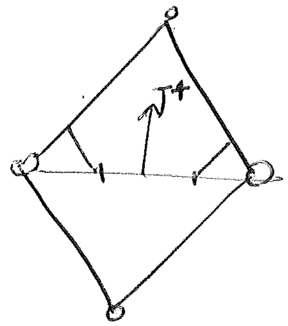
<u>3rd step:</u> show $\mathcal{R}(T^+)$ dense in $\mathcal{H}_0$.

Apply Hörmander's 1990.

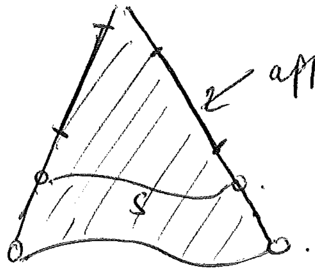
2nd case: (M,g). $M \simeq \mathbb{R}^4$.



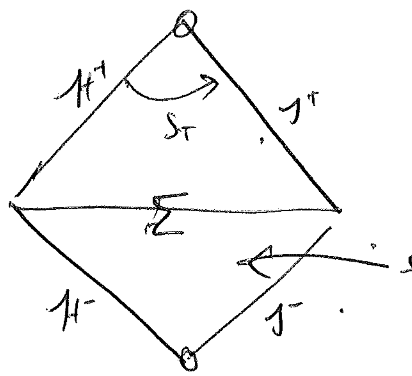
Cannot apply div. theorem as before, but use finite prop. speed:



Step 1 & 2 are the same, . Step 3:



3rd case: Schwarzschild.



Uniform estimates b/w

$S^+_T \cup \mathcal{H}^+_T \cup S_T$.

and Σ. prove that $\mathcal{E}_{S_T} \to 0$.

$T \to +\infty$.