Lecture 7

23/09/2014

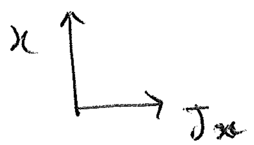
Correction: Last time.

$$R(u, \bar u, v, \bar v) = R(x, y, y, x) + R(x, Jy, Jy, x).$$

$$u = \frac{1}{\sqrt{2}}(x - iJx), \quad v = \frac{1}{\sqrt{2}}(y - iJy).$$

bisectional curvature of (u, v).

put $x = y \Rightarrow R(x, Jx, Jx, x)$ is the sectional curvature of the 2-plane $\{x, Jx\}$, so just the sectional curvature of the $\mathbb{C}$ 1-dim plane associated to x



Holomorphic sect. curvature assoc. to x.

Prove: (1) Const hol. sectional curvature then

$$R_{i\bar j k \bar l} = \lambda(g_{i\bar j} g_{k\bar l} + g_{i\bar l} g_{k\bar j}).$$

(w.r.t. an o.n. frame.

(2) holom. sect curv $\equiv$ cst.

$$\Rightarrow \tilde M \in \{ \underbrace{\mathbb{P}^n_{FS}, \mathbb{H}^n}_{\text{do not have const bisectional curvature.}}, \mathbb{C}^n_{Euc} \}.$$

(1) $\not\Rightarrow$ const bisectional curvature.

Holomorphic vector fields.

→ gauss complex dynamics'. want to study dynamics in $\mathbb{C}$-ceased, then need to preserve holomorphicity.

Ref: Kobayashi ~ '72, Gauduchon : Calabi's extremal metrics : an elementary intro.

(M, J) almost $\mathbb{C}X$.

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} + T^{0,1}.$$

$$x \longmapsto x^{1,0} = \frac{1}{2}(x - iJx).$$

Cauchy - Riemann operator:

E Cx v.b, rank $= r$.

$\downarrow$

(M, J).

$\bar{\partial}^E$ is a C.R. operator if it a first order op. in $C^\infty(E)$. w/ values in $E \otimes T^{0,1*}M$

$$\bar{\partial}^E(fs) = s \otimes \bar{\partial} f + f \bar{\partial}^E s, \quad f \in C^\infty(M), \ s \in C^\infty(E).$$

What is $\bar{\partial}$ since we only have (M, J) only al-Cx?

Connections ∇ connection on E, $\mathbb{C}$-linear.

By alc. Cx struct, $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

$$\nabla^{1,0}_x s := \frac{1}{2}(\nabla_x s - i\nabla_{Jx} s), \quad x \text{ real v. field.}$$

Check: $\nabla^{0,1}$ is a CR operator.

(E, h), h Hermitian metric; <u>hermitian vector bundle</u>.

∇ Hermitian if compatible with h.

$$x\, h(s_1, s_2) = h(\nabla_x s_1, s_2) + h(s_1, \nabla_x s_2)$$

for x any <u>Real</u> vectorfield.

~~Prop. $(E,h) \to (M,J)$ Hermitian v.b. Then, $\exists!$ Hermitian connection ∇ s.t. for any CR op $\bar\partial^E$, here $\bar\partial^E = \nabla^{0,1}$~~

<u>Prop</u> $(E, h) \longrightarrow (M, J)$. Hermitian v.b. Then, given a CR op. $\bar\partial^E$, there exists a unique Hermitian connection ∇ s.t. $\nabla^{0,1} = \bar\partial^E$.

<u>J integrable</u> $\Rightarrow$ Canonical CR operator assoc. to holomorphic v.b. (denote it by $\bar\partial^E$).

$\Downarrow$ Prop

$\exists$ canonical ∇^{Ch}, called Chern connection, associated to (E, J, h).

<u>Prop</u>: Let (M, g, J, ω) be an almost-Hermitian mfld.
[(M,J) al-cx, TM Cx v.b. assoc. to J, equip with Hermitian metric [illegible], ~~$h_{g,J}$ determines~~ ~~$h_{g,J}$~~ $h(x,y) = \frac{1}{2}[g(x,y) - i\omega(x,y)]$]

Thm, $\nabla^{Ch} = \nabla^{LC} \iff (M, g, J)$ kähler.

~~Rem~~

(M, J) cx, CR op on $TM \leftrightarrow T^{1,0}M$. $(X \to X^{1,0})$

$\bar\partial^{TM} = \bar\partial$

$$\bar\partial_Y Z = [Y^{0,1}, Z^{1,0}]^{1,0} \underset{\uparrow}{=} -\frac{1}{2} J(\mathcal{L}_Z J)Y.$$

Nijenhuis tensor $\equiv 0$.

$[Y^{0,1}, Z^{1,0}]^{0,1} = 0.$

Aside (Given Def^n: $X \in \Gamma(M, TM)$ (real) is a hol. v. field iff $\mathcal{L}_X J \equiv 0$.

eqv prop $\bar\partial X = 0$, where $\bar\partial$ CR op.)

~~Lemma: 2)~~ Assume (M, J) kähler.

Def^n: Denote $h(M, J)$ the v. space of hol. vec. fields.

Lemma 1) $h(M, J)$ is a cx Lie algebra.

2) ((M, g) cpct) $X \in h(M, J)$, then

$$X = X_H + \nabla f_1 + J\nabla f_2$$

$X_H = \alpha^\#$, $\alpha^\#$ g-harmonic 1-form.

f_1, f_2 $C^\infty(M)$ real valued.

α uniquely det upto

f_1, f_2 " " " constants.

1) $\mathcal{L}_X J = 0$, $\mathcal{L}_Y J = 0 \Rightarrow \mathcal{L}_{[X,Y]} J = 0$.

$([\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]})$.

$\mathcal{L}_X J = 0 \Rightarrow \mathcal{L}_{JX} J = 0$.

$\boxtimes$.

2) Harder.

<u>Lemma</u> $X \in h(M,J)$.

$\mathcal{L}_X \omega = i\partial\bar\partial f_1$, f_1 called potential to X.

~~Pf~~

$$\mathcal{L}_X \omega = \mathcal{L}_{X_H}\omega + \mathcal{L}_{\nabla f_1}\omega + \mathcal{L}_{J\nabla f_2}\omega.$$

$$\mathcal{L}_Y = d \circ i_Y + i_Y \circ d.$$

$$\Rightarrow \quad = d i_{X_H}\omega + d i_{\nabla f_1}\omega + d i_{\nabla f_2}\omega.$$

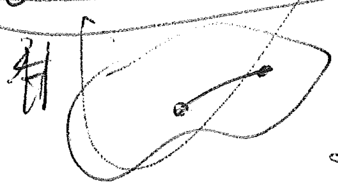
Hodge thm $\Rightarrow d i_{X_H}\omega = 0$.

$d i_{\nabla f_2}\omega = 0$ since $i_{J\nabla f_2}\omega = \pm d f_2$;

since $\omega(J\nabla f_2, Y) = -\omega(Y, J\nabla f_2)$
$= g(Y, \nabla f_2)$
$= -d f_2(Y)$

"Hol. v. fields behave like gradient vector fields". $\boxtimes$.

~~Exciting application~~



~~$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega + \sqrt{-1}\,\partial\bar\partial\varphi > 0\}$.~~

~~$X \in h(M,J)$, X induces an action on $\mathcal{H}$.~~

~~$\Phi(t) = \exp tX$. 1-para. of holo.~~

~~$\Phi(t)J = J$.~~

Exotic App

$$\mathcal{H} = \{ \varphi \in C^\infty(M) : \omega + \sqrt{-1}\,\partial\bar\partial\varphi > 0 \}.$$

$X \in h^0(M, J)$ reductive.

$\Phi_t = \exp tX$

$\Phi^*(t) J = J.$

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \omega = \mathcal{L}_X \omega = i\partial\bar\partial \theta_X.$$

$\Phi^*(t)\omega$.

path of Kähler metrics.