Lecture 6 18/09/2014.

Curvature in a Kähler manifold.

$$R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar z_l}\right) = g\left(R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_j}\right)\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar z_l}\right).$$

$$= R_{i\bar j k \bar l}.$$

R_{ijkl} usual Riemann. curv. tensor.

$TM \otimes_{\mathbb{R}} \mathbb{C}$ — cx rank $2n$ v. bundle on M.

$\underbrace{T^{1,0}M}_{\text{cx rank } n \text{ distribution}} \oplus T^{0,1}M$.

Kähler integrable $\Rightarrow$ integral distribution.
$([D,D] \subset D)$.

$e_1, \dots, e_{2n}$ real v. fields so that they span TM ~~(locally??)~~ (local o.n. frame).

$$X \longmapsto X^{1,0} := \frac{X - iJX}{2}.$$

$$X^{0,1} := \frac{X + iJX}{2}.$$

$$X = X^{1,0} + X^{0,1}.$$

$R_{ijkl} = R(e_i, e_j, e_k, e_l).$

$g(e_i, e_j) = \delta_{ij}$

$u_i = \frac{1}{\sqrt 2}(e_i - iJe_i).$

$u_{\bar i} = \frac{1}{\sqrt 2}(e_i + iJe_i).$

$\{u_1, \dots, u_n\}$ unitary basis for $T^{1,0}M$.

$$g(u_i, u_{\bar{j}}) = \frac{1}{2} \cdot g(e_i - iJe_i,\ e_j + iJe_j).$$
$$= \frac{1}{2}\left[\delta_{ij} + g(Je_i, Je_j)\right]. = \delta_{i\bar{j}}$$

since $g(e_i, Je_j) = 0$.

<u>Claim:</u> $Ric_{\mathbb{C}}(u_i, u_{\bar{i}}) = \# Ric(e_i, e_i)$.

$$\sum_j R(u_i, u_{\bar{i}}, u_j, u_{\bar{j}}).$$

$\& \ g^{k\bar{l}} R_{i\bar{i}k\bar{l}} = \delta^{k\bar{l}} R_{i\bar{i}k\bar{l}}$. since we choose o.n. basis.

$$Ric_{\mathbb{C}}(u_i, u_{\bar{i}}) = \frac{1}{4} Ric_{\mathbb{C}}(e_i - iJe_i,\ e_i + iJe_i,\ e_j - iJe_j,\ e_j + iJe_j)$$
$$= [R(e_i, e_i, e_j, e_j) + R(e_i, iJe_i, e_j, iJe_j) + \cdots].$$
$$= \sum_j - R(e_i, Je_i, e_j, Je_j).$$

$$Ric(e_i, e_i) = \sum_{j=1}^{n} R(e_i, e_j, e_i, e_j) + \sum_{j=1}^{n} R(e_i, Je_j, e_i, Je_j)$$

since $(e_1, \ldots, e_{2n}) = (e_1, \ldots, e_n, Je_1, \ldots, Je_n)$.

<u>Bisectional Curvature</u> ← Doesn't exist in Real world. (Lives b/w Ricci & Sectional's).

<u>Riem:</u> if x, y. $g(x,y) = 0$, $g(x,x) = g(y,y) = 1$,

The Sectional curvature $S(x,y) = R(x,y,y,x)$. of $\mathrm{span}\{x,y\}$.

$u = \frac{1}{\sqrt{2}}(x - iJx)$, $v = \frac{1}{\sqrt{2}}(y - iJy)$.

$R(u,\bar u,v,\bar v)$ = <u>bisectional</u> curvature associated to span$\{u,v\}$.

~~Thm~~ $R(u,\bar u,v,\bar v) = R(x,y,y,x) + R(x,Jy,Jy,x)$.

<u>Def^n</u> (M,J,g) Kähler has <u>const bisectional curvature</u> $= \lambda$ if

$$R_{i\bar j k\bar l} = \lambda(g_{i\bar j}g_{k\bar l} + g_{i\bar l}g_{k\bar j}).$$

<u>Ex.</u> $(\mathbb{C}^n, g_{Eucl})$ 0.

$(\mathbb{CP}^n, g_{FS})$ $+$.

(B, g_H) $-$. $B = \{z \in \mathbb{C}^n : |z| < 1\}$.

$\omega = i\partial\bar\partial \log(1 - \|z\|^2)$.

<u>Thm.</u> (M,J,g) complete & Kähler & const bisectional curvature.

Then M is one of these!

<u>Pf.</u> $M_\lambda \to M$, M_λ covering space ($\mathbb{C}^n$, $\mathbb{P}^n$ or B).

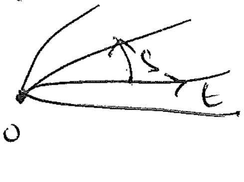
$$\begin{tikzcd} 0 \arrow[r, mapsto] & p \\ T_0M_\lambda \cong \mathbb{R}^n \cong T_pM \arrow[u, "\exp_0"] \arrow[ur, "\exp_p"'] & \end{tikzcd}$$

$\psi := \exp_p \circ \exp_0^{-1}$ (locally).

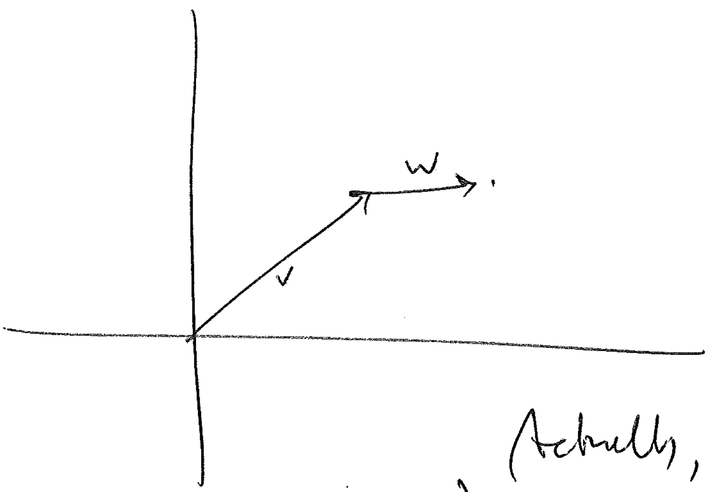
$d\psi =$ ~~$d\exp_p \circ d(\exp_0)$~~

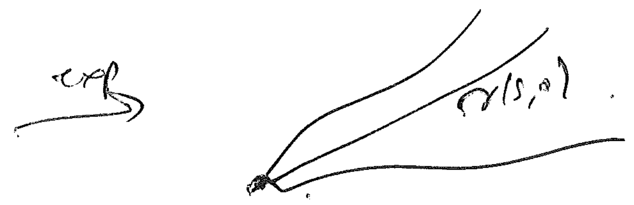
$d\exp_p \circ d(\exp_0^{-1})$.

$\gamma(s,t) = \exp_0(s(v + tw))$.

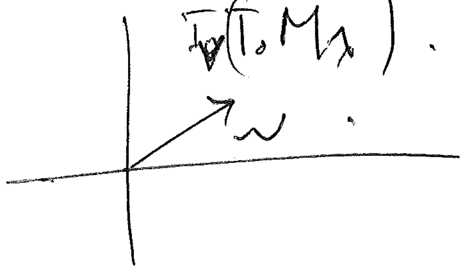


$T_0 M_\lambda$.





Actually, double tangent space.



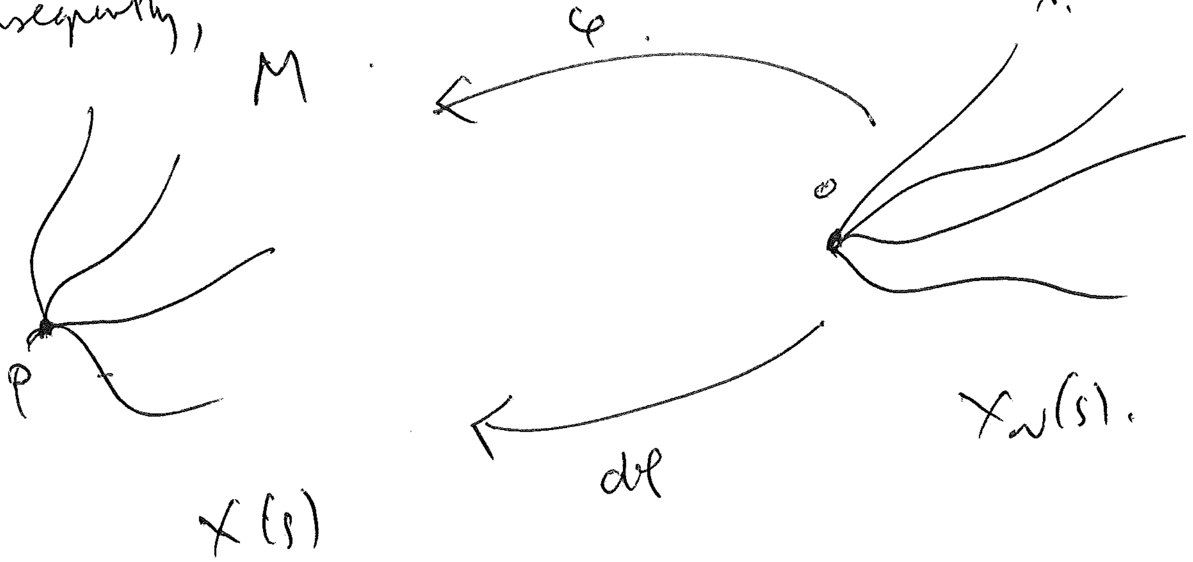
$$d\exp_0\big|_v : T_v(T_0 M_\lambda) \to T_{\exp_p(v)} M.$$

$$d\exp_p\big|_v(w) = \frac{\partial}{\partial t}\Big|_{t=0} \gamma(s,t) =: X_w(s).$$

So, $X_w(s)$ is a Jacobi field, i.e., solves:

$$\boxed{\nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} X_w = R\left(\frac{\partial \gamma}{\partial s}, X_w\right)\frac{\partial \gamma}{\partial s}, \quad X_w(0) = 0, \quad X_w'(0) = w}$$

Consequently,



Sol^{n}s are unique!

$e_1, \ldots, e_n, Je_1, \ldots, Je_n$ basis for $T_0 M_1$

$e_1 = \frac{\partial \gamma}{\partial t} / \left|\frac{\partial \gamma}{\partial t}\right|$. parallel along curve $\gamma(s,0)$.

$$\begin{cases} \nabla_{\frac{\partial \gamma}{\partial s}} e_i(s) = 0. \\ e_i(0) = e_i \end{cases}$$

$X_w(s) = x^i e_i$ (

$\langle e_i, \ddot{x}^i e_i - x^j R(\frac{\partial \gamma}{\partial s}, e_j) \frac{\partial \gamma}{\partial s} \rangle = \langle 0, e_i \rangle.$

$0 = \ddot{x}^i - x^j R(\frac{\partial \gamma}{\partial s}, e_j, \frac{\partial \gamma}{\partial s}, e_i)$ for any $i = 1, \ldots, 2n$.

$= \ddot{x}^i - x^j R(e_1, e_j, e_1, e_i) \left|\frac{\partial \gamma}{\partial s}\right|^2.$

Lemma $R(e_1, e_j, e_1, e_i)$ are determined by the bisectional curvature. (L113, Tian).