Lecture 5 UMD 868. 16/09/2014.

Projective Geometry.

$\mathbb{C}^{n+1}$ consider the space of all cx-dim. vector subspaces.

$$G_{\mathbb{C}}(1, n+1) = \mathbb{P}^n \equiv \mathbb{CP}^n.$$

What's a Kähler metric on $\mathbb{P}^n$ look like?

Work in charts. $\mathbb{C}^{n+1}$ coords $z_0, \ldots, z_n$;

$$U_\alpha = \{z_\alpha \neq 0\}. \quad U_\beta = \{z_\beta\}.$$

& induced coordinates

$$\{t_\alpha^i\}_{i=1}^n = \left\{\frac{z_0}{z_\alpha}, \ldots, \frac{z_\alpha}{z_\alpha} = 1, \ldots, \frac{z_n}{z_\alpha}\right\}.$$

Similarly, $\{t_\beta^i\}_{i=1}^n = \left\{\frac{z_0}{z_\beta}, \ldots, \frac{z_\beta}{z_\beta} = 1, \ldots, \frac{z_n}{z_\beta}\right\}$

transition on $U_\alpha \cap U_\beta$: multiplication by $\frac{z_\alpha}{z_\beta} = g_{\alpha\beta}$.

$$\varphi_{Euc} = |z_0|^2 + \cdots + |z_n|^2.$$

$$\cancel{\pi} \cdot i\partial\bar{\partial}(\varphi_{Euc}) = ?$$

on U_α : $i\partial\bar{\partial}\left(\left|\frac{z_0}{z_\alpha}\right|^2 + \cdots + \left|\frac{z_n}{z_\alpha}\right|^2\right).$

on U_β : $i\partial\bar{\partial}\left(\left|\frac{z_0}{z_\beta}\right|^2 + \cdots + \left|\frac{z_n}{z_\beta}\right|^2\right).$

$g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ $\quad f_\beta = f_\alpha \underbrace{\left|\frac{z_\alpha}{z_\beta}\right|^2}_{|g_{\alpha\beta}|^2}.$

So, $i\partial\bar{\partial} f_\beta|_{U_\alpha \cap U_\beta} \neq i\partial\bar{\partial} f_\alpha|_{U_\alpha \cap U_\beta}.$

On the other hand;

$g_{\alpha\beta}$ non zero on $U_\alpha \cap U_\beta$ holomorphic.

$$\log f_\beta = \log f_\alpha + \underset{\ker \partial\bar\partial}{\underset{\parallel}{\text{Pluriharmonic}}}.$$

So, $\Rightarrow i\partial\bar\partial \log f_\beta = i\partial\bar\partial \log f_\alpha$.

Lemma $\underbrace{i\partial\bar\partial \log\left(|z_0|^2 + \dots |z_n|^2\right)}_{\text{choice of metric, really } \pi_{*}\,g_{Euc}}$. $\quad \pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n$. $\quad \cong g_{Euc} \circ \pi^{-1}$.

is a globally defined (1,1)-form on $\mathbb{P}^n$.

Line Bundles: $M = \bigcup_\alpha U_\alpha$, ~~$U_\alpha = \{\varphi_\alpha\}_\alpha$~~.

where $\varphi_\alpha : U_\alpha \to \mathbb{R}$. function on U_α.

and $$i\partial\bar\partial \varphi_\alpha = i\partial\bar\partial \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

Side (1). Consider $\{h_\alpha := e^{-\varphi_\alpha}$.

remark

In our example, $\varphi_\alpha = -\log\left(\left|\frac{z_0}{z_\alpha}\right|^2 + \dots + \left|\frac{z_n}{z_\alpha}\right|^2\right)$.

$e^{-\varphi_\alpha} = \left|\frac{z_0}{z_\alpha}\right|^2 + \dots + \left|\frac{z_n}{z_\alpha}\right|^2$, $\quad g_{\alpha\beta} = \frac{z_\alpha}{z_\beta}$,

$h_\alpha |g_{\alpha\beta}|^2 = h_\beta$.

~~$g_{\alpha\beta} = \frac{\varphi_\alpha}{\varphi_\beta}$ (i.e. transition)~~.

$g_{\alpha\beta}$ – transition functions on M,

$$\left.\begin{array}{l} g_{\alpha\beta}\cdot g_{\beta\alpha} = 1 \\ g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \end{array}\right\} \Longrightarrow \{g_{\alpha\beta}\} \in H(M, \mathcal{O}^*).$$

Side remark 2

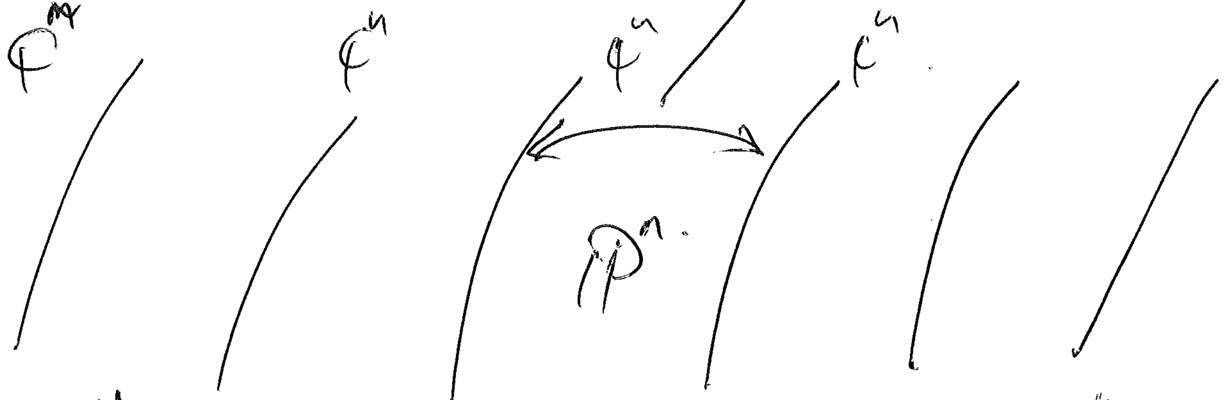
~~the~~ Čech-cohomology.

~~One back to line bundles: $[\text{illegible}]_\alpha := e^{-2\pi i\alpha}$ corresponds to an isomorphism class of line bundle.~~

$\{g_{\alpha\beta}\}$ corresponds to isomorphism class of a line bundle.

Back to our example:

multiply fibers by the $g_{\alpha\beta}$.



holomorphic section of line bundle. $s = \{s_\alpha\}_\alpha$.

$s_\alpha : U_\alpha \longrightarrow \mathbb{C}$. holomorphic funct.

s.t. $\quad s_\alpha = g_{\alpha\beta} s_\beta$.

<u>Example</u> (our example) ~~$s_\alpha = z_\alpha$~~ Fix γ.

$s_\alpha = \dfrac{z_\alpha}{z_\gamma}$ on ~~[illegible]~~ on U_α.

Back to good line bundle:

$\cancel{h_\beta |S_\alpha|^2 = h_\alpha |S_\beta|^2}$

<u>Steps</u>: (clarify):

~~① Start with . $S_\alpha = g_{\alpha\beta} S_\beta$.~~ <u>Confusing</u>.

~~② . take $h_\beta |g_{\alpha\beta}|^2 = h_\alpha$. , ~~choose~~~~

<u>Curvature of K-mfld</u>: christoffel symbols.

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma_{ij}^k \frac{\partial}{\partial z_k} + \underset{0}{\cancel{\Gamma_{ij}^{\bar{k}}}} \frac{\partial}{\partial \bar{z}_k}$$

$$\nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j} = \underset{0}{\cancel{\Gamma_{\bar{i}j}^k}} \frac{\partial}{\partial z_k} + \cancel{\Gamma_{\bar{i}j}^{\bar{k}}} \frac{\partial}{\partial \bar{z}_k}$$

since J is <u>parallel</u>. $\rightsquigarrow$ $\nabla_{\frac{\partial}{\partial \bar{z}_i}} (J \frac{\partial}{\partial z_j}) = \nabla_{\frac{\partial}{\partial \bar{z}_i}} (i \cdot \nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j})$

$$J \nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j} + \underset{0}{\cancel{(\nabla_{\frac{\partial}{\partial \bar{z}_i}} J)}} \frac{\partial}{\partial z_j}$$

$\Rightarrow$ (1,0)-type.

$\Rightarrow$ Γ_{ij}^k are the only relevant christ. symbs.

$$\boxed{\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k}}.$$

* Thinking of g ~~as~~ as a Hermitian metric

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(J\frac{\partial}{\partial z_i}, J\frac{\partial}{\partial z_j}\right) = 0.$$

Since $\overset{4}{-g(\text{———})}$.

So, $$\frac{\partial}{\partial z_i}\, g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar z_k}\right) = g\left(\Gamma^l_{ij}\frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar z_k}\right).$$

$$= g_{l\bar k}\,\Gamma^l_{ij}\, g^{s\bar k}.$$

$$= \delta^s_l\,\Gamma^l_{ij} = \Gamma^s_{ij} = g^{s\bar k} \underset{\partial_i g_{j\bar k}.}{\underset{\shortparallel}{g_{j\bar k;i}.}}$$

$$R(\partial_i, \partial_{\bar j}, \partial_k, \partial_{\bar l}) = g\left(\cancel{\nabla_i \nabla_{\bar j}}^{\,0}\,\partial_k - \nabla_{\bar j}\nabla_i\,\partial_k\,;\,\partial_{\bar l}\right)$$

$$= g\left(-\nabla_{\bar j}\left(\Gamma^t_{ik}\,\partial_t\right), \partial_{\bar l}\right).$$

$$= -\Gamma^t_{ik,\bar j}\, g_{t\bar l}.$$

check? $\boxed{= -\left(g^{t\bar m} g_{k\bar m, i}\right)_{\bar j} \cdot g_{t\bar l} = R_{i\bar j k\bar l}}$.

<u>Ricci</u>: $R_{i\bar j k\bar l}\, g^{i\bar j} =: Ric_{k\bar l}$

$$= g^{i\bar j} g_{tl}\left(g^{t\bar m} g_{k\bar m, i}\right)_{\bar j}$$

$$= -\left(\log\det[g_{i\bar j}]\right)_{k\bar l}.$$

derivatives in $\bar j$ and (undl) respectively.

$$\left(\frac{(\det[g_{i\bar j}])_k}{\det[g_{i\bar j}]}\right)_{\bar l} = \left(\frac{M_{ij}\, g_{i\bar j, k}}{\det[g_{i\bar j}]}\right)_{\bar l}$$

(M_{ij} = "minor" i, j th row removed from matrix)

$$\boxed{g^{i\bar j} = \frac{M_{i\bar j}}{\det[g_{i\bar j}]}}$$

$$\Rightarrow \left(g^{i\bar j} g_{i\bar j,k}\right)_{\bar l} = g^{i\bar j} g_{i\bar j,k\bar l} - g^{i\bar t} g^{s\bar j} g_{s\bar t,\bar l}\, g_{i\bar j,k}.$$

and compute ...

Back to ~~for~~ $\mathbb{CP}^n$ example:

$$i\partial\bar\partial \log\left(|z_0|^2 + \dots + |z_n|^2\right).$$

$$= i\partial\bar\partial \log\left(1 + |z_1|^2 + \dots + |z_n|^2\right).$$

in $\frac{z_\alpha}{z_0}$ coordinates.

$$g_{i\bar j} \overset{?}{=} ?$$

$$\partial\left(\frac{\sum z_j\, d\bar z_j}{1+\|z\|^2}\right) = (1+\|z\|^2) \dots$$

$$[g_{i\bar j}] = \frac{1}{(\|z\|^2+1)^2}\left[I(\|z\|^2+1) - z\bar z^T\right], \quad \text{eigenval } \{1, (1+\|z\|^2), \dots, (1+\|z\|^2)\}.$$

$$\det[g_{i\bar j}] = \frac{1}{(\|z\|^2+1)^{2n}} (1+\|z\|^2)^{n-1}.$$

$$\log\det[g_{i\bar j}] = \cancel{-(n+1)(1+\|z\|^2)}$$
$$= -(n+1)\log(1+\|z\|^2).$$
$$= -(n+1)\log(h).$$

$$R_{k\bar l} = (n+1)(\log h)_{k\bar l} = (n+1)\, g_{k\bar l}. \quad \text{I.e.,}$$

Ricci $= n+1$, so Kähler-Einstein.