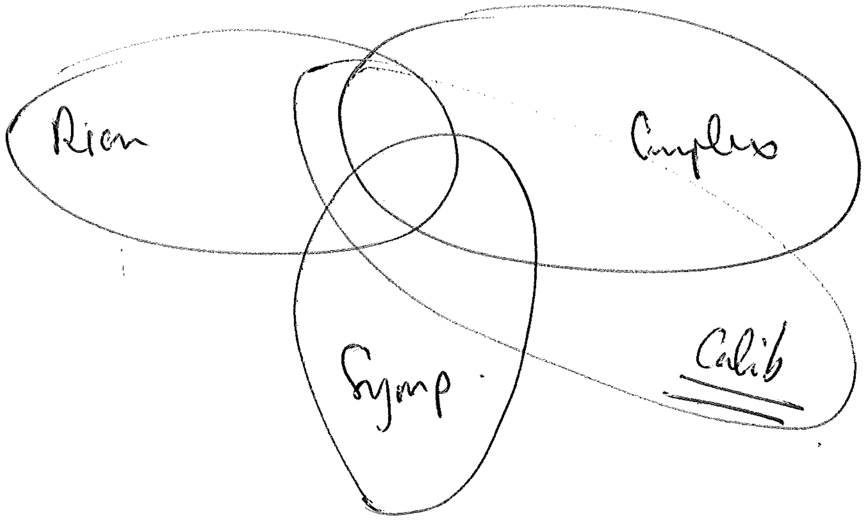
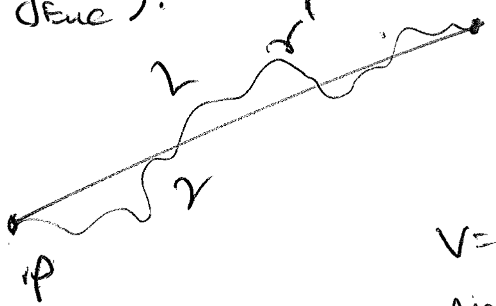
Lecture 4 UMS 11/09/2014

Calibrated geometry.



Calibration studied by Harvey–Lawson 1982.

$(\mathbb{R}^n, g_{euc})$.



lines minimise length.

V = unit const coeff vector parallel to L.

$$\text{length}(\gamma) = \int_\gamma (\text{dual 1-form to } V) = \int_\gamma V^\flat.$$

flat isom.

Let γ' (curved one) b/w p and q.

γ and γ' are homologous since we fix p and q.

$$\boxed{\text{length}(\gamma) = \int_{\gamma'} V^\flat}$$ b/c $V^\flat$ is closed.

$$\int_\gamma v^\flat - \int_{\gamma'} v^\flat = \int_{\gamma - \gamma'} v^\flat = \iint_{\mathcal{M}} \underbrace{dv^\flat}_{=0}.$$

$$\text{length}(\gamma') = \int_{\gamma'} Y^\flat \qquad \underline{Y = \dot\gamma}. \qquad |\dot\gamma| = 1.$$

$$\int_{\gamma'} v^\flat = \int_{\gamma'} v^\flat(Y) y^\flat = \int g(v, Y) Y^\flat.$$
$$\leq \int_{\gamma'} \underbrace{|v|_g}_{1} \underbrace{|Y|_g}_{1} Y^\flat.$$

Now, instead, consider $(\mathcal{M}, g)$;

let v be a $\underline{\text{killing field}}$ $\mathcal{L}_v g = 0$, $|v|_g \equiv 1$.

generates isometries, infinitesimal isometry.

$\alpha := v^\flat$, $\underline{d\alpha = 0}$. crucial (this is where the devil lives in! Devil $\equiv$ Energy).

$$d\alpha(Y, Z) = Y\alpha(Z) - Z\alpha(Y) - \alpha([Y, Z]).$$
$$= Yg(v, Z) - Zg(v, Y) - g(v, [Y, Z]).$$
$$= \cancel{g(\nabla_Y v, \nabla_Y Z)} + g(\nabla_Y v, Z) - g(\nabla_Z v, Y)$$
$$- \cancel{g(v, \nabla_Z Y)}.$$

claim $\to = 0$

If v killing $\Rightarrow$ ∇v is skew-symm. (Ex)

Ex. V killing $\Rightarrow$ $-\frac{1}{2}|\nabla|V|_g^2|^2 = \nabla_V V$.

But $|V|_g^2 = 0 \Rightarrow \nabla_V V = 0$.

So, integral curves of V are geodesics!.

Alt. pf that geodesics are length minimising?

Calibrations (finally).

~~Defn~~. Ref: Harvey, Spinors & Calibrations.

HL. 1982.

oriented p-dim subspace of $\mathbb{R}^n$

with $\xi = e_1 \wedge \dots \wedge e_p \in \Lambda^p \mathbb{R}^n$.

where $\{e_i\}$ o.n. basis for subspace.

$$G(p,n) = G(p,\mathbb{R}^n) = \{\xi \in \Lambda^p\mathbb{R}^n : \xi = e_1 \wedge \dots \wedge e_p, \text{ where } \{e_i\} \text{ o.n. basis}\}.$$

$(\Lambda^p\mathbb{R}^n)^*$ p-forms.

Defn p-form φ on $U \subset \mathbb{R}^n$ is a

calibration if

(I) $d\varphi = 0$

(II) $\forall x \in U$, $\varphi_x = \varphi(x) \in (\Lambda^p\mathbb{R}^n)^*$.

$\varphi_x(\xi) \leq 1$, $\forall \xi \in G(p,n)$.

$M \subset \mathcal{U}$ is a p-dim submfld s.t.

$\varphi_x(T_x M) = 1$.

Then M calibrated by φ.

In prev example, $p = 1$, $\varphi = \alpha$, L calibrated.

<u>Def^{n}.</u> M be closed, oriented p-submfld of $\mathbb{R}^n$,

M vol. mini if $\mathrm{vol}(M) \leq \mathrm{vol}(V)$.

where $\mathcal{U}$ rel. open subset of M w/ C^∞ bdy.

and V is cpct, oriented submfld. with $\partial V = \partial \mathcal{U}$.

$\boxed{\text{(Obs: Can be more } V = \emptyset \ \ M = M \text{ only. Need } [M] \neq [V] \text{ (homology?))}}$

<u>Th^m</u> (H-L).

φ calibrates M. Then any p-submfd calibrated by φ is volume minimizing.

<u>Pf</u>. $\mathrm{vol}(M) = \int_M \cancel{\varphi\, \mathrm{dvol}_M (T_x M)} = \int_V \varphi$.

$= \int_V \varphi(\vec{v})\, \lambda_{\vec{v}}$

$\vec{v} \in \Lambda^p TM$. $\lambda_{\vec{v}} \in (\Lambda^p TM)^\vee$.

φ calib $\Rightarrow \varphi(\vec{v}) \leq 1$.

$\Rightarrow \leq \int_V \lambda_{\vec{v}} = \mathrm{vol}(V)$. ∎

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Kähler setting:

Claim: (M, g, J) Kähler $\Rightarrow$ ω is a calibration.

Condition (I) in def^n true by Kähler. I.e, $d\omega = 0$ always.

(II) $n=1$. $\omega(x, Y) = g(Jx, Y)$
$\leq |Jx|_g |Y|$
$= |x|_g |Y| \leq 1$.

$Jx \parallel Y \Rightarrow \{x, Y\}$ $\mathbb{C}$ 1-dim space.

Th^m (Wirtinger's th^m).

$$\frac{1}{p!}\omega^p(\xi) \leq 1. \quad \forall \xi \in G_{\mathbb{R}}(p, 2n).$$

with eq. iff $\xi \in G_{\mathbb{C}}(p, n)$.

When is v. space $\mathbb{C}$? $JV = V$.
$JV \cap V = \{0\}$ if $\{\xi\}$ totally $\mathbb{R}$ $\}$.

NOTE: Once you have Kähler, then lots of minimal submanifolds! Every $\mathbb{C}$ submfld is Kähler!

Pf. Lemma. Let α be a 2-form on P, P is a v. space of dim $2p$. $\lambda_1, \ldots, \lambda_p > 0$. and $\theta_1, \ldots, \theta_{2p}$ an o.n. basis for P^* s.t. $\alpha = \lambda_1 \theta_1 \wedge \theta_2 + \ldots + \lambda_p \theta_{2p-1} \wedge \theta_{2p}$.

~~$\lambda_1 \theta_1 \wedge \theta_2 + \dots + \lambda_p \theta_{2p-1} \wedge \theta_{2p}$.~~

Back to Pf. ξ defines a v. space ~~of~~ P.

of $\dim_{\mathbb{R}} 2p$. $\frac{1}{p!} \omega^p \Big|_P (e_1 \wedge \dots \wedge e_{2p})$

where $\{e_1, \dots e_{2p}\}$ o.n. for P.

Using formula for Lemma for $\omega|_P$:

(write $\frac{1}{p!} \omega^p|_P = \frac{1}{p!} (\omega|_P)^p$)

$$\omega|_P = \sum_{i=1}^{p} \lambda_i \theta_{2i-1} \wedge \theta_i .$$

2-forms commute! So.

$$\frac{1}{p!} (\omega|_P)^p = \lambda_1 \cdots \lambda_p \, \theta_1 \wedge \dots \wedge \theta_{2p}.$$

So, $\frac{1}{p!} (\omega^p)|_P (e_1 \wedge \dots \wedge e_{2p}) = \lambda_1 \dots \lambda_p$.

But $\lambda_i = \omega_P | (\underbrace{e_{2i-1} \wedge e_{2i}}_{\text{from a } \mathbb{C} \text{ line}}) \leq 1$ (by 1-dim case).

eq $\Leftrightarrow$ $\lambda_i = 1$. $\forall i \Leftrightarrow \xi = e_1 \wedge Je_1 \wedge \dots \wedge e_p \wedge Je_p$.
$\in G_{\mathbb{C}}(p,n)$.

Pf of Lemma. $F(x \wedge y) = \alpha(x \wedge y)$, $F : G_{\mathbb{R}}(2, 2n) \to \mathbb{R}$.

Sps x, y achieve max of F. $\circledast$

Claim $\alpha(x \wedge z) = 0$ $\forall z \perp y$. $\lambda_1 = F(x \wedge y)$, $\theta_1 = x$,

$\theta_2 = y$ and induct on $\alpha - \lambda_1 x \wedge y$. on $\{x, y\}^{\perp}$.

Consider $f(\theta) = F(x \wedge \cos\theta y + \sin\theta z)$. $f'(0) = 0 = \sin\theta \lambda_1 + \cos\theta$ $F(x \wedge z)|_0 = 0$.
variation of y.