Lecture 3   09/09/2014.

$\partial\bar\partial$ - lemma

Last time: local $\partial\bar\partial$-lemma on $\mathbb{C}^n$.
$\Rightarrow$ α (1,1)-form closed $\Rightarrow$ $\alpha = i\partial\bar\partial f$.
$\exists f \in C^\infty(\mathbb{C}^n)$.

Note: this allows to work with functions rather than 2-forms. So locally, $\omega = g(J\cdot, \cdot)$ is reduced to functions.

Kähler mfld: cpct, no bdy. $\rightsquigarrow$ "closed"

Goal: global $\partial\bar\partial$-lemma.

M closed, kähler.

Lemma If ω is (1,1) form on M and $d\omega = 0$, and then $\tilde\omega$ is also closed & (1,1) and $[\omega] = [\tilde\omega]$, then $\tilde\omega = \omega + i\partial\bar\partial f$ where $f \in C^\infty(M)$.

If α is closed ($d\alpha = 0$), then $\alpha \longmapsto [\alpha] \in H^*(M, \mathbb{R})$

I.e., de Rham cohomology class of α.

$\boxed{\text{Another way of saying this: } \alpha \text{ is (1,1)-form and d-exact, then } \alpha \text{ is } \partial\bar\partial\text{-exact.}}$

Dolbeault cohomology theory $\rightsquigarrow$ $\partial, \bar\partial$ operators.

de Rham cohomology.

d operator. $0 \to C^\infty(M) \xrightarrow{d} \Omega^1(M) \to \quad \to 0$.

de Rham cohom. gives finite dimensionality
of certain associated spaces → ie exact, closed etc.
This is via estimates.

Similarly, in the Dolbeault case:

$H^*_{\bar\partial}(M, \mathbb{C})$, $H^*_{\partial}(M, \mathbb{C})$.

ie $\Omega^{0,q}(M) \xrightarrow{\partial} \Omega^{1,q}(M) \xrightarrow{\partial} \dots$

These things are usually diff, but in Kähler,
they are equal.

$\Delta = dd^* + d^*d$, $\langle d\alpha, \beta\rangle_g := \langle \alpha, d^*\beta\rangle_g$.

$\boxed{* \text{ w.r.t. } g}$

$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$.

$\Delta_\partial = \partial\partial^* + \partial^*\partial$.

Kähler: $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$. (F)

~~This says the~~
Std Hodge: $\alpha = H\alpha + d\beta$,
But fact (F) means that $N(\Delta_\partial) = N(\Delta_{\bar\partial}) = N(\Delta)$!

Pf (Griffiths-Harris).

$d\alpha = 0 \implies (\partial + \bar\partial)\alpha = 0$;

$\implies \begin{cases} \partial\alpha = 0 \\ \bar\partial\alpha = 0. \end{cases}$

$\alpha = H\alpha + \Delta_{\bar\partial} G\alpha$

$\quad = H\alpha + \bar\partial\bar\partial^* G\alpha + \cancel{\bar\partial^*\bar\partial G\alpha}$

assuming α is d-exact $\Rightarrow$ $\underline{H\alpha = 0}$.

$\bar\partial \Delta_{\bar\partial} = \Delta_{\bar\partial}\bar\partial$ so it commutes with G.

So, $\alpha = \bar\partial\bar\partial^* G\alpha$.

α type $(1,1)$, $G\alpha$ type $(1,1)$, $\bar\partial^* G\alpha$ type $(1,0)$

$\partial\bar\partial G\alpha = \pm \bar\partial^* G(\partial\alpha) = 0$.

$\Rightarrow$ $\bar\partial G\alpha$ ∂-closed. $(1,0)$ form.

$\bar\partial G\alpha = \cancel{H'\alpha} + \partial\partial^* G(\partial^* G\alpha) + \cancel{\partial^*\partial G(\bar\partial^* G\alpha)}$.

again by exactness. Same G, since Laplacians are the same.

So, $\alpha = \bar\partial\partial\partial^* G(\bar\partial^* G\alpha) = -\partial\bar\partial(\underbrace{\partial^* \underbrace{G \underbrace{\bar\partial^* \underbrace{G\alpha}_{(1,1)}}_{(1,0)}}_{(1,0)}}_{(0,0)})$

Implications:

We define, given a Kähler metric (form) ω, we have a recipe to construct many other K metrics on M: Choose any $f \in C^\infty(M)$, $t > 0$ small enough, $\omega_t = \omega + 2t\,\partial\bar\partial f$ is K metric.

This is positive form $t>0$ small and type $(1,1)$ means that it is compatible with J. So $\Rightarrow$ ω_t Kähler form.

Conversely, by $\partial\bar\partial$-lemma, any Kähler form $\tilde\omega$ satisfying $[\tilde\omega]=[\omega]$ is of the form

$$\tilde\omega = \omega + i t\,\partial\bar\partial f.$$

This kind of parametrisation of metrics is not possible with Riemannian setting $\rightsquigarrow$ too many directions.

$$g + t\nabla f(\nabla f)^T \qquad \text{rank } 1.$$
$$e^F g \qquad \text{conformal} ,$$

etc.

Def^n The space of Kähler form cohomology to ω is

$$\mathcal{H}_\omega = \{\varphi\in C^\infty(M) : \underline{\omega + i\partial\bar\partial\varphi > 0}\} \subset C^\infty(M).$$

~~open in~~

- $\mathcal{H}_\omega$ is open in $C^\infty(M)$ and in neighborhood of 0. In fact $\mathcal{H}_\omega$ is convex.

$\alpha\in(1,1)$ form positive means $\alpha(v,Jv)>0$ $\forall$ real v.

Comparing to Riem setting: Space of Riem metrics is really quite nasty!

properties of ∞-dim. space $\mathcal{H}_\omega$. $\Leftrightarrow$ properties of M.
$\mathcal{H}_\omega(\mathbb{P}^1)$ is not well understood!

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, $\mathbb{P}^1 = \{$all complex lines ~~through~~ in $\mathbb{C}^2\}$.
$\cong S^2 \cong \mathbb{C}^2/\mathbb{C}^* \cong S^3/S^1$. Hopf.

Now, take $\mathbb{P}^1$ in ~~homo. coordinates~~.
homogeneous coordinates $[x:y]$.
Embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ $[x:y] \mapsto [x^2 : xy : y^2]$.
derivative x and y of map.
$[2x : y : 0]$, $[0 : x : 2y]$, so nonzero derivative.

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ via $[x:y] \mapsto [x^N : x^{N-1}y : \dots : xy^{N-1} : y^N]$.

[Natural metrics in $\mathbb{P}^1$: Fubini-Study metric on projective space: Arising as quotient.]

So, embedding pullback induces metric.

Remarkable result: $\mathcal{H}_\omega(\mathbb{P}^1)$ has a dense subset which is the union of all the metrics obtained via pullback letting $N \to \infty$ (Analogue to Stone-Weierstrass)

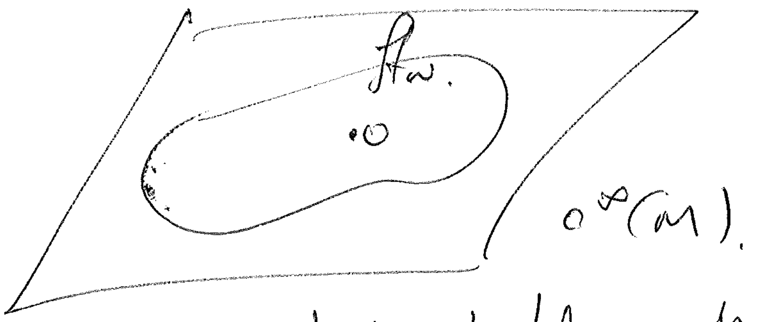
① Fubini-Study metrics.
② submanifold geometry for Kähler.
③ analogue for general M (line bundle & holom. sections).
④ Approximation statement: $\mathcal{H}_\omega$ approx by algebraic objects.

i.e., $\mathcal{H}_\omega$ approximation via alg. objects is a connection to Algebraic Geometry.

(II). Relation to geometric PDEs:

Mabuchi metric on $\mathcal{H}_\omega$:

$T_\varphi \mathcal{H}_\omega = C^\infty(M)$.



$\langle h, k \rangle|_\varphi = \int_M hk \, \underbrace{dV_{g_\varphi}}$

volume associated to Kähler metric ω_φ.

Geodesics of $\langle \cdot, \cdot \rangle$ $\rightarrow$ complex Monge-Ampère eq^n.

conjectural asymptotic to Ricci flow.!

<u>My note:</u> Has anyone studied Wasserstein space of $\mathcal{H}_\omega$?