Lecture 2 UMD 868.

~~g R~~

g Riem, J comp.

$$\omega(X,Y) = g(JX,Y).$$

~~(M, ω, g, J)~~

Def^n (M,ω,g,J) Kähler structure, when $d\omega = 0$ (1),

- ~~dω = 0~~ (1) says that ω is symplectic.

~~(1)~~
- (1) $\Rightarrow$ $T^{1,0}$ is invariant under g-parallel translations.

$N_J \equiv 0$ and $d\omega = 0$ $\Leftrightarrow$ $\nabla J = 0$.

$$d\omega(x,y,z) = x\,\omega(y,z) + y\,\omega(z,x) + z\,\omega(x,y)$$
$$- \omega([x,y],z) - \omega([y,z],x) - \omega([z,x],y).$$
$$= x\, g(Jy,z) + y\, g(Jz,x) + z\, g(Jx,y).$$
$$- \quad \cdots$$
$$= g(\nabla_x(Jy),z) + g(Jy,\nabla_x z).$$
$$+ \cdots$$

from metric compatibility.

By $(\nabla_x J)y = (\nabla_x Jy) - J\nabla_x y$.

$= \nabla_x(Jy) - J\nabla_y x - J[x,y]$.

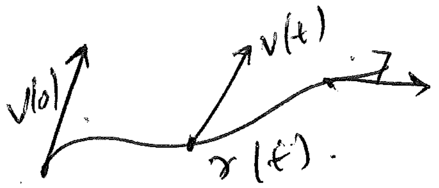
we get

$d\omega(x,y,z) = g((\nabla_x J)y, z) + g((\nabla_y J)z, x) + g((\nabla_z J)x, y)$.

So, $\nabla J = 0 \Rightarrow d\omega = 0$.

$d\omega(x,y,z) - d\omega(Jx, Jy, z) = g(z, JN_J(x,y)) + 2g((\nabla_z J)x, y)$.

$\nabla J = 0 \Rightarrow$ parallel trans. preserves type ?



<u>Assume</u> $Jv(0) = i v(0)$.

<u>Claim:</u> $Jv(t) = iv(t)$.

parallel means $\nabla_{\dot\gamma} v(t) = 0$ (so by assumption).

$\nabla_{\dot\gamma}(Jv(t)) = 0$. ∀

<u>Why?</u>

$\parallel$

$(\nabla_{\dot\gamma} J)v(t) + J(\nabla_{\dot\gamma} v)$.

2) $\nabla J = 0 \Leftrightarrow \nabla\omega = 0$.

$(\nabla_x\omega)(y,z) = \nabla_x(g(Jx, z)) - \omega(\nabla_x y, z) - \omega(y, \nabla_x z)$.

$= \xrightarrow{\text{metric compt.}}$

$= g((\nabla_x J)y, z)$.

Holonomy.

$Hol_x(M,g)$.

- parallel transport v along loop γ.
- parallel transport preserves angle.

$v(1) = A v(0)$.

$A \in GL(2n, \mathbb{R})$.

- But A must preserve type, so A not general. rather: $A \in GL(n, \mathbb{C})$.

$$A + iB \in GL(n, \mathbb{C}) \Rightarrow \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in GL(2n, \mathbb{R}).$$

Also $A \in O(2n)$. So

$$A \in GL(n, \mathbb{C}) \cap O(2n) \Rightarrow A \in U(n) \text{ (unitary)}.$$

So, $Hol_0(M,g) \subset U(n)$.

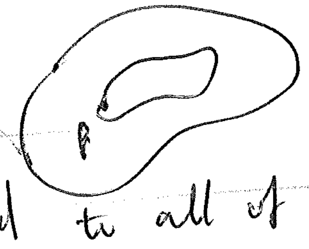
<u>In fact</u>: $Hol(M,g) \subset U(n) \Rightarrow g$ kähler.

(to be careful this means $\forall p \in M$. $Hol_p(M,g) \subset U(n)$).

<u>Pf</u>

$\gamma : T_pM \xrightarrow{P_\gamma} T_pM$. $P_\gamma \in U(g(p))$.

$g(v,w) = g(P_\gamma v, P_\gamma w)$, $\nabla g = 0$; $P_\gamma \in O(T_pM)$.



$z_1, \dots, z_n$

$w(p) = P_{\gamma} dz^i \wedge d\bar{z}^i$ check?

extend to all of M by parallel transport.

$\nabla \omega = 0$ and $\omega^n \neq 0$, since $\nabla(\omega^n) = 0$ which is false.

at p, define $J_p : T_pM$ ⟲ by

$\omega(v,w)|_p = g(Jv, w)|_p$.

$J^2|_p = -I$; $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$.

Extend J by p to M. (by ⟶)

$\nabla(J^2 + I = 0) \Rightarrow \nabla J = 0$ So (M, g, J) Kähler. ∎

Typically: $Hol \subset O(\dim_{\mathbb{R}} M)$.

K $\iff$ $Hol \subset U\left(\frac{\dim_{\mathbb{R}} M}{2}\right)$.

Calabi-Yau $\iff$ $Hol \subset SU(n)$.

$\partial\bar{\partial}$ Lemma   Hodge theory.

Prop (Local $\partial\bar{\partial}$ lemma).

(M, J) cx and ω is $(1,1)$ form defined on contract. open set U s.t. $d\omega|_U = 0$. Then locally

$\exists f \in C^\infty(M)$ s.t. $\omega|_U = -i\partial\bar{\partial}f$.

Pf: $d\omega = 0$, $d = \partial + \bar{\partial}$, $\partial\omega + \bar{\partial}\omega = 0$ (diff types) $\Rightarrow$ ① $\partial\omega$, ② $\bar{\partial}\omega = 0$.

Dolbeault cohom (of $\bar{\partial}$) on $\Phi^n(U)$ is trivial.

$\omega = \bar\partial\alpha$, α type $(1,0)$.

plug into ① $\quad \partial\bar\partial\alpha = 0$.

$\quad$ '' C.R.

$\bar\partial(\partial\alpha)$

$\partial\alpha = \bar\partial\beta \implies \alpha = \partial f$, so $\omega = \bar\partial\partial f$,

f is imaginary, so mult. by i to get real f.

<u>Note:</u> This is purely local, so always true on ~~Can~~ Cx mfld. <u>Point:</u> on Kähler, this can be improved to global.