Lecture 21 20/11/2014

Atiyah-Singer index Thm.

M cpct, oriented, dim M divisible by 4, $\not{\Delta}M$ normed spinor bundle and

$$i(\not{D}_M := \not{\Delta}^+ \to \not{\Delta}^-) = \left(\frac{i}{2\pi}\right)^m \int_M \langle \hat{A}(R), d\rho \rangle \qquad m = n/2$$

$\hat{A}$ roof functional on R, Reimann curvature tensor.

Remark: $\hat{A}$ exists outside of this context.
The integral $\left(\frac{i}{2\pi}\right)^m \int_M \langle \hat{A}(R), d\rho \rangle$ non-integer precisely means that M does not admit a spin-structure.
In fact, $\hat{A}$ was discovered first, and $\not{D}_M$ was discovered during the investigation as to why the integral is sometimes integer!

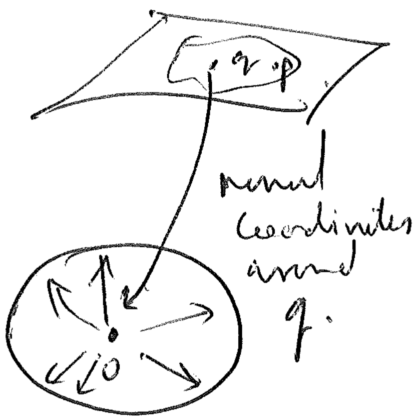
$$i(\not{D}_M : \not{\Delta}^+ \to \not{\Delta}^-) = \frac{1}{(4\pi)^m} \int_M \underbrace{\left(\operatorname{Tr}\left(H^m(q,q)\big|_{\not{\Delta}^+}\right) - \operatorname{Tr}\left(H^m(q,q)\big|_{\not{\Delta}^-}\right)\right)}_{= 2^m i^m (\overline{\omega}_n(q) H^m(q,q))\big|_{\Delta^0}} dq.$$

Recall $H^k(p,q) = \sum_{\beta=0}^{\infty} H^k_\beta(p,q)$. and that.

$H^k_\beta(p,q) \in W_{2k+\beta}$, $\quad W_j = \Delta^0\mathbb{R}^n \oplus \Delta^1\mathbb{R}^n \oplus \dots \oplus \Delta^j\mathbb{R}^n$.

and mod $W_{2k+\beta}$, have recursion formula

$$(\beta+k)H^k_\beta = D_2 H^{k-1}_{\beta+2} + D_1 H^{k-1}_\beta + D_0 H^{k-1}_{\beta-2}.$$



$$D_1 = -\frac{1}{2} \sum_{ij} x_i R_{ij} \partial_j$$

$$D_0 = -\frac{1}{16} \sum_{ijk} x_i R_{ij} R_{jk} x_k.$$

$$D_2 = \Delta.$$

$R_{ij} = R_{ij}(q)$, curvature bivector at q.

$H^0_0 = 1$, $H^0_i = 0$, $i \geq 1$, initial condition.

Further simplification: D_1 can be commuted.

$$H^m_0 = \sum D_{i_1} \underset{\substack{\| \\ D_1}}{D_{i_2}} \cdots D_{i_m} 1.$$
mod W_{2m-1}.

$$[D_1, D_2] = [x_i R_{ij}\partial_j ; \partial_k^2]$$
$$= \underbrace{[x_i, \partial_k^2]}\, R_{ij}\partial_j.$$

$$\underbrace{[x_i,\partial_k]}_{\delta_{ik}}\partial_k + \partial_k \underbrace{[x_i,\partial_k]}_{\delta_{ik}} = \partial_i.$$

$$= R_{ij}\partial_i\partial_j = 0 \quad \left(\text{since } \begin{matrix} R_{ij} = -R_{ji} \\ \partial_i\partial_j = \partial_j\partial_i \end{matrix}\right)$$

Also, $[D_1, D_0] = 0$ "mod $W_{2k+\beta-1}$".

<u>Example</u>. dim $M=4$, simplest non-trivial case.

$$H_0^2 = \frac{1}{2} D_2 H_2' = \frac{1}{2} \cdot \frac{1}{3} D_2 D_0 1 = \frac{1}{6} \Delta\left(-\frac{1}{16} \sum_{ijk} x_i R_{ij} R_{jk} x_k\right).$$

$$= -\frac{1}{6(16)} R_{ij} R_{jk} \underbrace{\Delta(x_i x_k)}_{2\delta_{jk}}.$$

$$= -\frac{1}{3 \times 16} R_{ij} \wedge R_{jk}.$$

↑ diff modules.

But only interested in 4 vector part, so

its $\frac{-1}{3\times 16} R_{ij} \wedge R_{jk}$ we're interested in, and

$$i(D\!\!\!/_M : \Delta^+ \to \Delta^-) = \left(\frac{1}{4\pi}\right)^2 2^2 \cdot i^2 \int \sum_{ij} \left(\frac{-1}{3\times 16}\right) \langle R_{ij} \wedge R_{ji}, d\hat{p} \rangle.$$

$$= \frac{1}{192\pi^2} \int \Sigma_{ij} \langle R_{ij} \wedge R_{ji}, d\hat{p} \rangle.$$

Commutative alg $(\Lambda^{ev}\mathbb{R}^n, \wedge)$, so note that

$$\Sigma_{ij} R_{ij} \wedge R_{ji} = \mathrm{Tr}(R^2). \qquad R = (R_{ij}).$$

Model for general case of ASIT, replace $(\Lambda^{ev}\mathbb{R}^n, \wedge)$ by $(\mathbb{R}, \cdot)$, by $R_{ij} \longmapsto (A_{ij})$ skew.

Assume
$$A = \begin{pmatrix} \begin{matrix} 0 & a_1 \\ -a_1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & a_2 \\ a_2 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}.$$

Solve recursion. $(\beta+k) H^k_\beta = D_2 H^{k-1}_{\beta+2} + D_0 H^{k-1}_{\beta-2}$.

$D_2 = \Delta$, $D_0 = -\frac{1}{16} \sum_i \tilde{a}_i^2 x_i^2$.

(matrix looks like $\begin{pmatrix} -a_1 a_1 & & \\ & -a_2 -a_2 & \\ & & \ddots \end{pmatrix}$).

$\tilde{a}_{2j} = \tilde{a}_{2j-1} = a_j$.

Backwatch.

This recursion comes from solving

$$\partial_t f = \underbrace{\left(\sum_i \partial_i^2 + \left(\frac{\tilde{a}_i}{4}\right)^2 x_i^2\right)}_{D^2} f \qquad (1)$$

$f_t : \mathbb{R}^n \to \mathbb{R}$.

$$e^{-tD^2} f = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} \sum_{k=0}^{\infty} t^k \underbrace{H^k(x,y)}_{\substack{\text{need} \\ H^{2k}(0,0)}} f(y)\, dy.$$

$f(t,x) \in \mathbb{R}$.

$$\partial_t f = -\underbrace{\left(-\frac{d^2}{dx^2} + x^2\right)}_{\text{+ve op } L} f. \qquad \text{Harmonic oscillator.}$$

Mehler's formula.

$$(2) \quad e^{-tL} f = \frac{1}{\sqrt{2\pi \sinh(2t)}} \int_{\mathbb{R}} \exp\left(\frac{-\cosh(2t)\left(\frac{x^2+y^2}{2}\right) + xy}{\sinh(2t)}\right) f(y)\, dy.$$

Rescale and replace x_k by $i\frac{\tilde{a}_k}{2}x_k$ to explain (1) by (2).

Evaluate kernel at $x=y=0$.

$$e^{tL}f(x) = \sqrt{\frac{i\,\tilde{a}_k/4}{2\pi \sinh\left(2i\frac{\tilde{a}_k}{4}t\right)}}$$

So, over diff dimensions, kernel becomes product.

$$\prod_k \sqrt{\frac{\tilde{a}_k/2}{4\pi \sin\left(\frac{\tilde{a}_k}{2}t\right)}} = \frac{1}{(4\pi)^{m}t^{m}}\sum_{j=0}^{\infty} t^j H^j(0,0).$$

$$\parallel$$

$$\prod_k \frac{a_k/2}{4\pi \sin\left(\frac{a_k}{2}t\right)}$$ since $\tilde{a}_{2k}$, $\tilde{a}_{2k+1}$ repeats a_k.

So,

$$\frac{a_1 t/2}{\sin\left(\frac{a_1 t}{2}\right)} \cdots \frac{a_m t/2}{\sin\left(\frac{a_m t}{2}\right)}$$

$$= \sum_{j=0}^{\infty} t^j H^j(0,0).$$

Def^n $f(z) = \prod_k \frac{z_k/2}{\sin(z_k/2)}$.

$P_0(z) :=$ m-homogeneous part of Taylor exp. of f.

symmetric.

comm. alg. tells you this can be written by elementary poly's.

$$P_0(z_1, \ldots, z_m) = p(t_1, \ldots, t_{m/2}), \quad t_j := 2(-1)^j \sum_{k=1}^{m} z_k^{2j}$$

$$\hat{A}_0^m = P_0(a_1, \ldots, a_m) = \ldots = \underline{p\left(\mathrm{Tr}(A^2), \mathrm{Tr}(A^4), \ldots, \mathrm{Tr}(A^m)\right)}$$

invariant under coord transfos, and if you change matrix to non-block diagonal A, and also for curvature bivectors R_{ij}.

<u>Def^n</u>. $\hat{A}(R) := p\left(\mathrm{Tr}(R^2), \mathrm{Tr}(R^4), \ldots, \mathrm{Tr}(R^m)\right)$.

$$\mathrm{Tr}(R^k) = \sum R_{i_1 i_2} \wedge R_{i_2 i_3} \wedge \ldots \wedge R_{i_k i_1}.$$