Lecture 20 17/11/2014.

M cpct Riemannian mfld, oriented and admits a spin structure, $\dim M = n = 2m$.

ΔM Clifford bundle.
$\not\Delta M$ normal spinor bundle.

Fibrewise action. $\Delta M \xrightarrow{\sim} L(\not\Delta M)$, $\langle \cdot, \cdot \rangle_{x}$ & $\dagger$ on $\not\Delta M$.

Concretely, ON. frame $\{e_S\}_{S \in \mathcal{S}}$ for ΔM. $\mathcal{S} = \{-m, \ldots, -1, 1, \ldots, n\}$.

Induced ON-frame for $\not\Delta M$. $\{\not e_S\}_{S \in \not{\mathcal{S}}}$ for ΔM. $\not{\mathcal{S}} = \{1, \ldots, m\}$.

Just on a nbhd, so needs frame for TM, ΔM. $\mathcal{U} \subset M$.

L.C. connections.

TM: $\langle w_{ij}, v \rangle$.

ΔM: $\nabla_v e_S = \cancel{\tfrac{1}{2}[\omega^2, e_S]}\ \frac{1}{2} [\omega^2(v), e_S]$.

$$\omega^2(v) = \sum_{i<j} \langle w_{ij}, v \rangle e_i e_j = \frac{1}{2} \sum_{ij} \langle w_{ij}, v \rangle e_i e_j$$

$\not\Delta M$ existence of L.C. connection:

$$\nabla_v \not e_S := \frac{1}{2} \omega^2(v) \cdot \not e_S.$$

Verify that this satisfies prop. 1-3.

(2) duality :
(3) conjugation :) Real bivector : ω^2, so this is trivial.

(1) Need to ~~check~~ compute $\nabla_v e_i \phi_s$.

But induced ϕ_s is given by :

$$\begin{tikzcd} \overset{e_i}{\Delta_p M} \arrow[r] & \Delta_p M \cdot \phi_t \\ \overset{\bar{e}_i}{\Delta \mathbb{R}^n} \arrow[u, "\text{isom.}"] \arrow[r] & \Delta \mathbb{R}^n \arrow[u, "\bar{\phi}_s"] \end{tikzcd}$$

indices unique (mod sign)

I.e., action $e_i \cdot \phi_t$ is "like" $\bar{e}_i \cdot \bar{\phi}_s$.

Need $\nabla_v (\underbrace{e_t \cdot \phi_s}_{\pm \phi_{s'}}) = (\nabla_v e_t) \cdot \phi_s + e_t \cdot \nabla_v \phi_s$.

But $\nabla_v (e_t \cdot \phi_s) = \nabla_v \phi_{s'} = \frac{1}{2} \omega^2(v)(e_t \cdot \phi_s)$.

$\nabla_v \phi_s = e_t \cdot \frac{1}{2} \omega^2(v) \cdot \phi_s$

Commutation formula: $(\nabla_v e_t) \cdot \phi_s = \frac{1}{2} (\omega^2(v) e_t - e_t \omega^2(v)) \cdot \phi_s$

$= \frac{1}{2} \omega^2(v) e_t \cdot \phi_s - e_t \, \omega^2(v) \phi_s$.

Def^n. Spin / A-S Dirac operator on ΔM is :

$$D\!\!\!/_M \psi := \sum_{i \in K} e_i \cdot \nabla_{e_i} \psi .$$

Rem. $D\!\!\!/_M$ acts as a skew-adjoint op. in $L^2(\Delta M)$.

Sub-bundles.

Recall from before $\Lambda M = \Lambda^0 M \oplus \dots \oplus \Lambda^n M$.

But we don't need this grading, rather $\Lambda M = \Lambda^{od} M \oplus \Lambda^{ev} M$.

Recall that $D_M : \Lambda^{od} M \to \Lambda^{ev} M$.

$$\not{\Delta} M = \not{\Delta}^{ev} M \oplus \not{\Delta}^{od} M = \not{\Delta}^{+} M \oplus \not{\Delta}^{-} M.$$

and $\not{D}_M : \not{\Delta}^+ M \to \not{\Delta}^- M$.

Def^n ($\not{\Delta}^{\pm} M$.)

Denote by w_n the microbundle of M. I.e.,

$w_n \in C^\infty(M, \Lambda^n V)$, $|w_n| = 1$.

$$w_n^2 = w_n \wedge w_n = (-1)^{\frac{n(n-1)}{2}} \overline{w_n} w_n = (-1)^m \underbrace{\overline{w_n w_n}}_{=1}.$$

~~n~~ m even!

And $(i^{-m} w_n)^2 = +1$, so let

$\not{\Delta}^+ M$ be the $+1$-eigenspace of $i^{-m} w_n$.

$\not{\Delta}^- M$ be the -1-eigenspace of $i^{-m} w_n$.

H/W. $\not{D}_M$ swaps $L^2(M, \not{\Delta}^+ M)$ and $L^2(M, \not{\Delta}^- M)$.

Index problem: compute the index.

$$i(\not{D}_M : \Delta^+M \to \Delta^-M) = \int_M \langle ?, d\hat{p} \rangle.$$

<u>The Heat eqn method.</u>

$$i(\not{D}_M : \Delta^+ \to \Delta^-) := \dim N(\not{D}_M|_{\Delta^+}) - \dim N(\not{D}_M|_{\Delta^-})$$

$\downarrow \not{D}_M|_{\Delta^+}$ normal $\quad (\not{D}_M|_{\Delta^+})^*$.

$$= \dim N(\not{D}_M^2|_{\Delta^+}) - \dim N(\not{D}_M^2|_{\Delta^-}).$$

$$= \mathrm{Tr}\,(e^{t\not{D}_M^2}|_{\Delta^+}) - \mathrm{Tr}\,(e^{t\not{D}_M^2}|_{\Delta^-}).\ \forall t>0.$$

$\underline{t \to 0} \Rightarrow\ = \frac{1}{(4\pi)^m}\int_M (\mathrm{Tr}\,H^m(q,q)|_{\Delta^+}$

$$- \mathrm{Tr}\,H^m(q,q)|_{\Delta^-})\, dq.$$

where $(e^{t\not{D}_M^2} g)(p) \approx \int_M \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(p,q)}{4t}} \sum_{k=0}^{m} t^k H^k(p,q) g(q)\, dq.$

$H^0(q,q) = I \rightsquigarrow H^0(p,q)$

and then through recursion/induction

$H^1(p,q), \ldots, H^m(p,q).$

Weitzenböck formula for $\not{D}_M$.

$$\not{D}_M^2 \psi = \Delta_{\not{S}M}\psi - \sum_{i<j} e_i e_j \cdot \left(\langle \Omega, e_i \wedge e_j \rangle \psi \right).$$

$$\Delta_{\not{S}M}\psi = \sum \nabla_{e_i}(\nabla_{e_i}\psi) - \nabla_{\nabla_{e_i} e_i}\psi.$$

Ω: curvature for $\not{S}M$.

Difference / novelty is in $\langle \Omega, e_i \wedge e_j \rangle \psi$, the rest is the same as before.

Prop 12.24 gives: $\langle \Omega, e_i \wedge e_j \rangle = \frac{1}{2} R^2(e_i \wedge e_j)\psi$.

where $R^2(b) = \sum_{k<l} \langle R_{kl}, b \rangle \, e_k e_l$.

R_{kl}: Reimann bivector for TM.

So, $$\not{D}_M^2 \psi = \Delta_{\not{S}M}\psi - \frac{1}{8}\sum_{ijkl} e_i e_j R_{ijkl} e_k e_l \cdot \psi$$

$$= \Delta_{\not{S}M}\psi - \frac{1}{8} \underbrace{\sum_{ijkl} R_{ijkl} e_i e_j e_k e_l}_{= 2S \text{ (} S \text{: Scalar curv)}} \psi$$

$$= \Delta_{\not{S}M}\psi - \frac{1}{4} S \cdot \psi.$$

Clifford expression for traces

Isomorphism $\Delta M \cong \mathcal{L}(\not{S}M)$.

$$2^m \, T\big|_{\Delta^0} = \mathrm{Tr}(T) \quad \text{in} \quad \Delta M.$$

$$\Rightarrow \mathrm{Tr}(T|_{\not{S}^+}) - \mathrm{Tr}(T|_{\not{S}^-})$$

leaves $\not{S}^\pm$ invariant. (because we're thinking $\not{D}_M^2$, which leave invariant, unlike $\not{D}_M$ which swaps.)

$$= 2^m \left(i^{-m} w_n T\right)\Big|_{\Delta^0} \quad \text{(via isomorphism).}$$

Remark: $i^{-m} w_n$ looks like $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$.

wrt T looks like $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

The difficulty as compared to CGB thm..

- Need to solve recursion. $H^0 \rightarrow H^1 \rightarrow \dots \rightarrow H^m$.
  Steps: m.

- Need to multiply with at least $n = 2m$ vectors in total to reach the Δ^n section starting from Δ^0. $\left(i^{-m} w_n T\right)\big|_{\Delta^0} \simeq T\big|_{\Delta^n}$.

- In each step, need 2-vectors to multiply with.
  vs. CGB, only needed to multiply with
  4 vectors and didn't need to solve ODE.
  $\to$ cannot ignore derivative terms in the ODE.

- Fix $q \in M$, normal coordinates, $p = \rho(x)$, $q = \rho(0)$.
- $\{e_i\}$ ON-frame for TM obtained via polar factorisation of coordinate frame.
- Induced frames for ΛM and ΔM.
- $w^2(e_i)(p) = \frac{1}{2}\sum_{kl} \langle w_{kl}, e_i\rangle e_k e_l = \underbrace{\frac{1}{2}\sum_{kl}\sum_j R_{ijkl}\, x_j}_{\substack{x=0,\ w^2=0,\ \text{as normal coords.} \\ \text{metric in terms of curvature,}}} + O(x^2, \Delta^2)$.

The ODE:

$$\left(\nabla_{r_q} + \frac{1}{4}\partial_{r_q}\ln(q) + k\right) H^k(p,q) = \not{D}_M^2\, H^{k-1}(p,q).$$

$$\Leftrightarrow_{(*)} \left[\sum_i x_i\left(\partial_i + \frac{1}{4}\sum_j R_{ij}\, x_j + O(x^2,\Delta^2)\right) + O(x^2,\Delta^0) + k\right] H^k$$

$$= \sum_i \left(\partial_i + \frac{1}{4}\sum_j R_{ij}\, x_j + O(x^2,\Delta^2)\right)\cdot$$

$$\left(\partial_i + \frac{1}{4}\sum_j R_{ij}\, x_j + O(x^2,\Delta^2)\right)$$

$$- \sum_{ii'} \langle \nabla_{e_i} e_{i'}, e_{i'}\rangle \left(\partial_i + \frac{1}{4}\sum_j R_{ij}\, x_j + O(x^2,\Delta^2)\right)$$

$$H^{k-1} - \frac{1}{4} S\, H^{k-1}.$$

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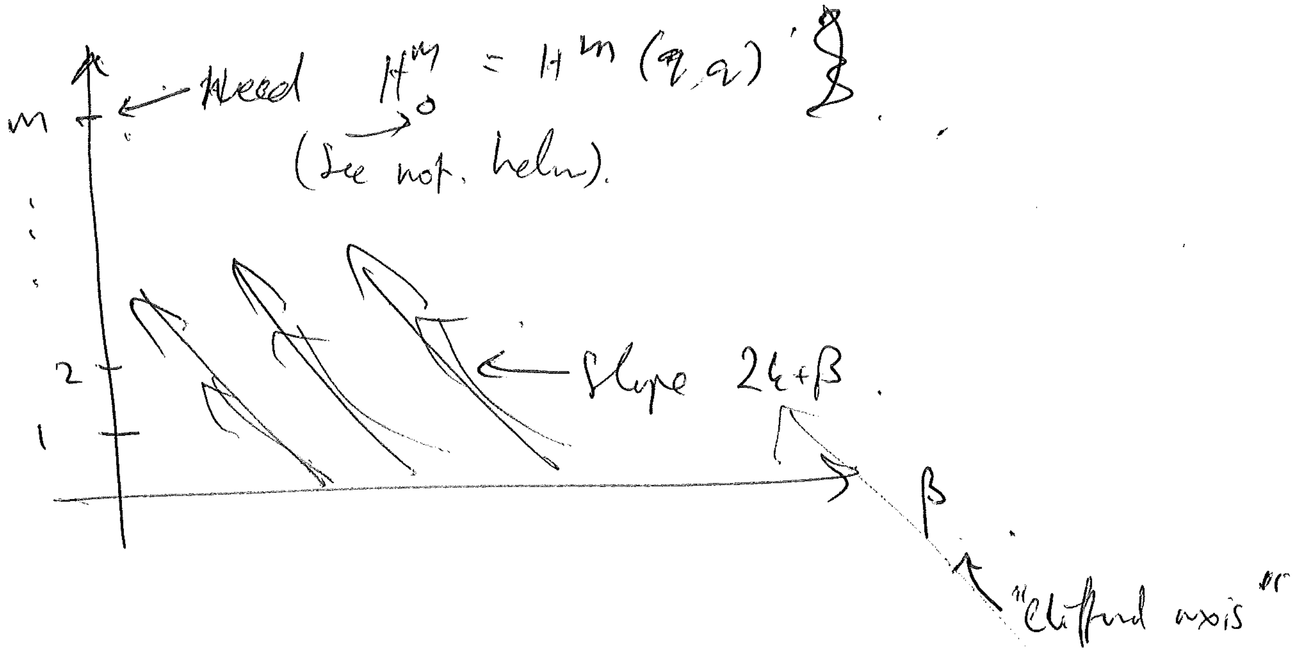

Set : $W_j := \Delta^0 M \oplus \Delta^1 M \oplus \cdots \oplus \Delta^j M$.

$$H^k(p,q) = \sum_{\beta=0}^{\infty} \underbrace{H^k_\beta(p,q)}_{\substack{\beta \text{ hom pol,} \\ \text{Taylor exp in } p}} \qquad \left(q \text{ fixed, fnct. of } p\right)$$

<u>Claim.</u> $H^k_\beta \in W_{2k+\beta}$,

Verify by induction over $2k+\beta$.
Calculate modulo $W_{2k+\beta-1}$.

Evaluate (*) modulo $W_{2k+\beta-1}$:

$$\underbrace{\sum_i x_i \partial_i H^k_\beta}_{\beta} + k H^k_\beta = \sum_i \partial_i^2 H^{k-1}_{\beta+2} + \frac{1}{2}\sum_{ij} R_{ij} x_j \partial_i H^{k-1}_\beta + \frac{1}{16}\sum_{i,j,i',j'} R_{i'j'} x_{j'} R_{ij} x_j H^{k-1}_{\beta-2}$$

$$D_2 = \Delta, \quad D_1 = \frac{1}{2} \sum_i R_{ij} x_j \partial_i .$$

$$D_0 = \frac{1}{16} \sum_{i j j'} R_{ij} x_j R_{ij'} x_{j'}$$

So, Recursion reads:

$$\beta H^k_{\beta+} + k H^k_\beta = D_2 H^{k-1}_{\beta+2} + D_1 H^{k-1}_\beta + D_0 H^{k-1}_{\beta-2} .$$