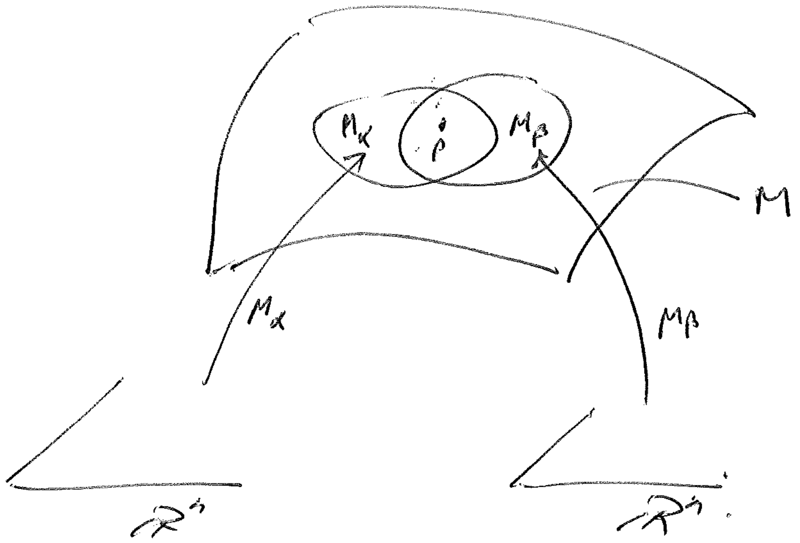
Lecture 19 13/11/2014.

Construction of Spinor bundles.

$M = \bigcup_\alpha M_\alpha$



Assume that $\{M_\alpha\}$ are good (ie., diffeomorphic to balls and finite intersections diffeomorphic to balls). Recall "goodness" is required for Čech cohomology to work.

TM = tangent bundle, no reason for $\underline{\mu}_\alpha(p)$ to be isometries. Polar decomposition.

$$\underline{\mu}_\alpha(p) = \underbrace{\hat{\underline{\mu}}_\alpha(p)}_{\substack{\text{isometry} \\ T_x\mathbb{R}^n \to T_pM}} \cdot \underbrace{G^{\frac{1}{2}}(p)}_{\text{Symmetric } T_x\mathbb{R}^n \to T_x\mathbb{R}^n}.$$

Replace $\underline{\mu}_\alpha$ by $\hat{\underline{\mu}}_\alpha$.

Problem. Can we choose the bundle charts so that $\hat{\underline{\mu}}_\alpha \circ \hat{\underline{\mu}}_\beta^{-1}$ are sense preserving. Ie, $\det \hat{\underline{\mu}}_\alpha \circ \hat{\underline{\mu}}_\beta^{-1} = 1$?

Use Čech ~~sto~~ cohomology for the sheaf $\mathbb{Z}_2 = \{0, 1\}$.

k-cochain: to each $(k+1)$-fold intersection,

$$M_S = M_{s_1} \cap \cdots \cap M_{s_{k+1}},$$

we have $\langle f, e_S \rangle \in \mathbb{Z}_2$.

<u>Define</u> Let f be the 1-cochain.

$$\underbrace{\langle f, e_{\alpha\beta} \rangle}_{M_\alpha \cap M_\beta} := \begin{cases} 0 \ ; & \text{if } \hat{M}_{\alpha\beta} \text{ sense preserving.} \\ 1 \ ; & \text{if } \det \hat{M}_{\alpha\beta} \leq -1. \end{cases}$$

$$\langle \partial_1 f, e_{\alpha\beta\gamma} \rangle = \sum_i^N \langle f, e_i \lrcorner e_{\alpha\beta\gamma} \rangle.$$

$$= \langle f, e_{\beta\gamma} \rangle - \langle f, e_{\alpha\gamma} \rangle + \langle f, e_{\alpha\beta} \rangle.$$

$$\equiv \langle f, e_{\beta\gamma} \rangle + \langle f, e_{\alpha\gamma} \rangle + \langle f, e_{\alpha\beta} \rangle.$$

Since in $\mathbb{Z}_2$, $-1 \equiv +1$.

We have $\det(\hat{M}_{\alpha\beta} \hat{M}_{\beta\gamma}) = \det \hat{M}_{\alpha\gamma}$, in $\mathbb{Z}_2$.

So, $\langle \partial_1 f, e_{\alpha,\beta,\gamma} \rangle \equiv 0$ and hence,

obtain a well defined Čech cohomology element:

$$N(\partial_1)/R(\partial_0) = \mathcal{H}^1 = [f] = w_1(M).$$

1st Stiefel-Whitney class.

Assume that $\tilde{\mu}_\alpha = \hat{\mu}_\alpha R_\alpha$, R_α is an isometry. (i.e. rotation or reverses orientation), and write $\tilde{f}$ for the corresponding 1-cochain.

Define 0-cochain g: ~~$\langle g, e_\alpha\rangle :=$~~

$$\langle g, e_\alpha\rangle := \begin{cases} 0 & \text{if } \det R_\alpha = 1. \\ 1 & \text{if } \det R_\alpha = -1. \end{cases}$$

$$\Rightarrow \quad \langle \partial_0 g, e_{\alpha\beta}\rangle = \langle g, e_\beta\rangle + \langle g, e_\alpha\rangle$$
$$= \langle \tilde{f}, e_{\alpha\beta}\rangle - \langle f, e_{\alpha\beta}\rangle.$$

$$\Rightarrow \quad \tilde{f} = f + \partial_0 g.$$

$$\Rightarrow \quad [\tilde{f}] = [f].$$

<u>Def^n.</u> M is orientable if $w_1(M) = 0.$ $(= [0])$.

$\big(w_1(M) \in H^1(M, \mathbb{Z}_2). \big)$.

Assume from here that M is orientable, and ~~define~~ that $\hat{\mu}_{\beta\alpha}(p) \in SO(\mathbb{R}^n)$. (orientability allows us to choose each $\hat{\mu}_{\beta\alpha}(p)$ to be a rotation).

Lecture 4 $\Longrightarrow$ $\exists\; q_{\beta\alpha}(p) \in Spin(\mathbb{R}^n). \subset \Delta^{ev}\mathbb{R}^n$.

So that $\hat{\mu}_{\beta\alpha}(p) V = q_{\beta\alpha}(p) \wedge V \wedge q_{\beta\alpha}(p)^{-1}$.

$q_{\beta\alpha}$ is unique up to sign ± 1.

Problem 2. Can $q_{\beta\alpha} \in Spin(\mathbb{R}^n)$ be chosen so that

$$\begin{cases} q_{\gamma\beta}(p) \cdot q_{\beta\alpha}(p) = + q_{\alpha\gamma}(p) \\ q_{\alpha\beta}(p) = q_{\beta\alpha}^{-1}(p). \end{cases} \quad ?$$

trivial to satisfy

transitivity is the issue.

Def^n. Let f be the 2-cochain

$$\langle f, e_{\alpha\beta\gamma} \rangle = \begin{cases} 0 \ ; & \text{when transitivity holds.} \\ 1 \ ; & \text{when it doesn't} \end{cases}$$

$M_\alpha \cap M_\beta \cap M_\gamma$.

$$= \begin{cases} 0 : & q_{\gamma\beta} q_{\beta\alpha} = + q_{\gamma\alpha}. \\ 1 : & q_{\gamma\beta} q_{\beta\alpha} = - q_{\gamma\alpha}. \end{cases}$$

Similar to above, ~~$\partial_2 f = 0.$~~

(1) $\partial_2 f = 0$

(2) for any other lift, ie choose $q.$ to either q or $-q$, $\tilde{q}_{\alpha\beta}$,

$\tilde{f} = f + \partial_1 g.$

I.e, $[\tilde{f}] = [f] =: W_2(M).$

2nd Stiefel-Whitney class.

Def^n If $w_2(M) = 0$, then we say M admits a spin structure.

Now assume that M ~~ha~~ admits a spin structure and is orientable, and <u>even dim.</u>

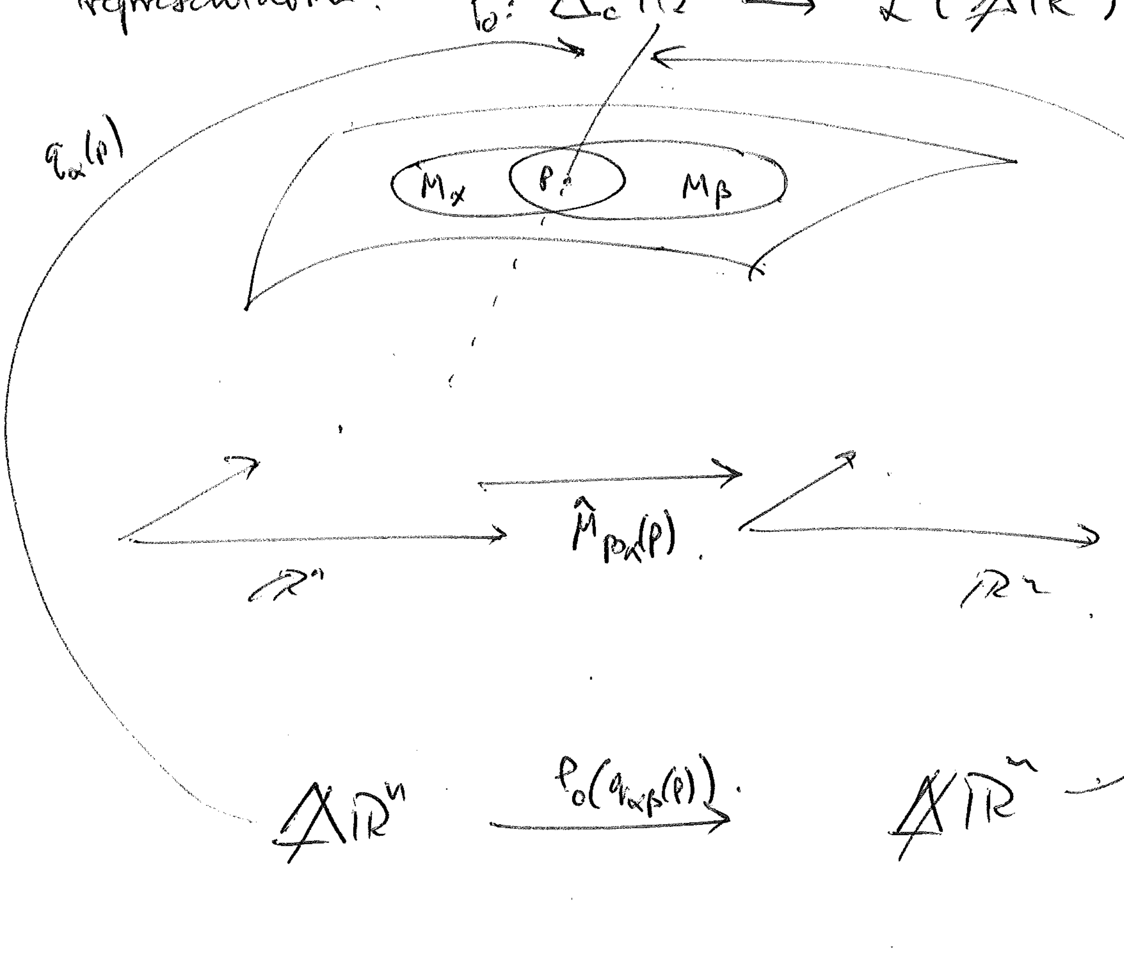
Fix such a transitive lift of transition maps.

$$\hat{\mu}_{\beta\alpha}(p)v = q_{\beta\alpha}(p)\, v\, q_{\beta\alpha}(p)^{-1}$$

Construct a spinor bundle over M denoted by $\not\Delta M$. ⟵ Complex vector bundle.

in even dim

Fix representation. $\rho_0 : \Delta_c \mathbb{R}^n \xrightarrow{\cong} \mathcal{L}(\not\Delta \mathbb{R}^n)$.



Define Fibre $\slashed{\Delta}_p M = \left\{ [\alpha, \psi] : \alpha \in \overline{N}, \psi \in \slashed{\Delta}\mathbb{R}^n \right\}$.

where $[\alpha, \psi] \sim [\beta, \varphi]$

if $\varphi = \rho_0(q_{\beta\alpha}(p))\,\psi$.

Bundle ~~transition~~ trivialisations:

This is why you want $w_2(M)=0$. $\rightarrow$ transitivity implies $\sim$ is an equivalence relation.

$$\left[\begin{array}{ll} q_\alpha(p) : & \slashed{\Delta}\mathbb{R}^n \longrightarrow \slashed{\Delta}_p M. \\ & \psi \longmapsto [\alpha, \psi]. \end{array}\right.$$

Define duality and conjugation on $\slashed{\Delta}M$:

$$\left.\begin{array}{l} (\mathrm{I})\ \langle [\alpha,\psi], [\alpha,\varphi] \rangle_* := \langle \psi, \varphi \rangle_* \\ (\mathrm{II})\ [\alpha,\psi]^\dagger := [\alpha, \psi^\dagger]. \end{array}\right\}$$ Fix std duality and conjugation in representation ρ_0.

Why indep of α?

$$\langle [\beta,\psi],[\beta,\varphi] \rangle = \langle \rho_0(q_{\beta\alpha})\psi, \rho_0(q_{\beta\alpha})\varphi \rangle_* = \langle \psi, \varphi \rangle_*$$

$$\shortparallel$$

$$\langle q_{\beta\alpha}\cdot\psi, q_{\beta\alpha}\cdot\varphi \rangle_*$$

$$\shortparallel$$

$$\langle \psi, \overline{q_{\beta\alpha}}^{c} q_{\beta\alpha}\varphi \rangle_* = \langle \psi, \underbrace{\overline{q_{\beta\alpha}} q_{\beta\alpha}}_{1}\varphi \rangle_* = \langle \psi, \varphi \rangle_*.$$

real, so lose $\mathbb{C}$-conj.

$$(q_{\beta\alpha}\psi)^{\dagger} = q_{\beta\alpha}\cdot(\psi^{\dagger}), \text{ so } (\mathrm{II}) \text{ is okay.}$$

So, we've constructed a mixed spinor bundle,

$$\langle \not{\Delta}M, \langle\cdot,\cdot\rangle_{\not{\Delta}}, \dagger \rangle.$$

<u>Note to self:</u> $\langle\cdot,\cdot\rangle_{\not{\Delta}}$ and $\dagger$ depend ~~cru~~ crucially on the metric, ~~be~~ because the factorisation ~~$\mu_{\alpha\beta} = \hat{\mu}_{\alpha\beta} G^{\frac{1}{2}}$~~ $\mu_\alpha = \hat{\mu}_\alpha G^{\frac{1}{2}}$ depends on metric, and the entire construction then depends on $\hat{\mu}_\alpha$. In fact "isometry" ~~co~~ properties depend on g and the geometry here lives in the charts.

<u>Levi-Civita covariant derivative on $\not{\Delta}M$.</u>

We require:

(1). $\nabla_v(F\cdot\psi) = (\nabla^{\Delta M}_v F)\cdot\psi + F\cdot(\nabla_v\psi)$.

$F \in C^\infty(\Delta M)$, $\psi \in C^\infty(\not{\Delta}M)$.

(2). $\partial_v\langle\psi,\varphi\rangle_{\not{\Delta}} = \langle\nabla_v\psi,\varphi\rangle_{\not{\Delta}} + \langle\psi,\nabla_v\varphi\rangle_{\not{\Delta}}$.

(3). $(\nabla_v\psi)^{\dagger} = \nabla_v(\psi^{\dagger})$. (Covariant derivative should be Real!)

Prop. $\exists!\ \nabla$ satisfies (I) – (III).

Pf. ! — assume ∇_v, ∇_v' two such connections.

$L_v \psi = \nabla_v' \psi - \nabla_v \psi$.

$$L_v(f\psi) = \partial_v f \psi + f\nabla_v'\psi - \partial_v f \psi - f\nabla_v \psi.$$
$$= \cancel{f\nabla_v'\psi}\ f\, L_v(\psi).$$

So, L_v is a linear operator on each fibre.

(I) Tells us that $L_v(F\psi) = F \cdot L_v(\psi)$. $F \in C^\infty(\Delta M)$.

$\boxed{\mathrm{II}}$ $\Rightarrow$ $(L_v \psi)(p) = \lambda(p)\, \psi(p)$. Since F can denote any matrix.

(II) Gives. $\langle \lambda \psi, \varphi \rangle_v = -\langle \psi, \lambda \varphi \rangle_v$.

$\cancel{\Rightarrow \operatorname{Im} \lambda = 0.}$ $\operatorname{Re} \lambda = 0$.

(III). Gives. $(\lambda \psi)^+ = \lambda(\psi^+) \cancel{\Rightarrow \operatorname{Im}\lambda = 0}$

$\lambda^c (\psi^+) = \lambda(\psi^+) \Rightarrow \operatorname{Im} \lambda = 0.$