Lecture 18 10/11/2014.

GB II Th^n.

We need to calculate:

$$(*)\quad \frac{1}{(4\pi)^m}\int_M \mathrm{Tr}\left(H^m(q,q)\big|_{\Lambda^{ev}}\right) - \mathrm{Tr}\left(H^m(q,q)\big|_{\Lambda^{odd}}\right) dq.$$

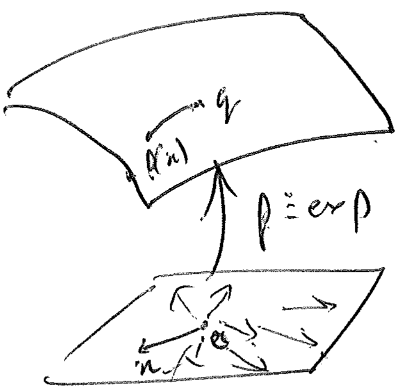
in terms of Riemann curvature R, $m = \frac{\dim M}{2}$, $q \in M$.

$H^k(p,q) : \Lambda_q M \to \Lambda_p M$ finite dim. lin. map.

which solves

$$p \mapsto \left(\nabla_{r_q} + \frac{1}{4}\partial_{r_q}\left((\ln g) + k\right)\right) H^k(p,q)$$
$$= D_M^2 H^{k-1}(p,q)$$

for $k = 0, \dots, m$.
and $H^0(q,q) = I$.



(radial vectorfield X, so
$\rho\, r_q = \rho\, P_* X$.)

If $p = q$, $H^k(q,q) = \frac{1}{k} D_M^2 H^{k-1}(p,q)\big|_{p=q}$.

Aim: To approximate D_M^2 well enough so that we can compute $(*)$

<u>Setup:</u> $\rho = \exp_q$ = normal chart.

$\{\bar{e}_i\}$ o.n. std basis for $\mathbb{R}^n$.

$\{\rho_*(\bar{e}_i)\}$ non-ON frame.

Polar decomposition: $\rho_* = \mathcal{U}S$; $\mathcal{U}$ - isometry, S symm.

$$\left(\text{I.e., } S = G^{\frac{1}{2}}, \quad g(u,v) = u^T G v\right).$$

$$\mathcal{U} : T_x\mathbb{R}^n \longrightarrow T_pM.$$

O.N. frame $e_i = \mathcal{U}(\bar{e}_i)$.

<u>Clifford algebras:</u> Recall $\mathcal{L}(\Lambda T_pM) \cong \Delta(T_pM)^2$.

$$(T_pM)^2 = T_pM \oplus T_pM$$

$$e_i^+ : w \longmapsto e_i \lrcorner w + e_i \wedge w \quad \in \mathcal{L}(\Lambda T_pM)$$

$$e_i^- : w \longmapsto -e_i \lrcorner w + e_i \wedge w \quad \in \mathcal{L}(\Lambda T_pM).$$

and $\Delta(T_pM)^n = \text{span}\,\{e_i^+, e_i^- : i = 1, \dots, m\}$.

Also, e_i^+, e_i^- anti-commute!

<u>Hanne rule:</u> $T \in \Lambda V \to \Lambda V$, then

$$\operatorname{Tr}(T) = 2^n \cdot T\big|_{\Lambda^0}.$$

$$\Rightarrow \quad \mathrm{Tr}\left(T|_{\Lambda^{ev}}\right) - \mathrm{Tr}\left(T|_{\Lambda^{od}}\right) = \mathrm{Tr}\left(T_{2^n} T\right)$$
$$= 2^n \left(T_{2^n} T\right)\Big|_{\Delta^0}$$

$$T_{2^n} = e_1^+ e_1^- e_2^+ e_2^- \cdots e_n^+ e_n^- \propto w \longmapsto \hat{w}.$$

(Assume T preserves Λ^{ev} and Λ^{od} respectively).

But $T_{2^n} T$ is almost $\propto T$, so

$$T_{2^n} T \simeq T|_{\Delta^{2n}}.$$

Now consider: $H^{k-1}(p,q) f(q) =: g(p)$.

Need $D_M^2 g$, recall from lecture 13 that:

$$\nabla_{e_i} e_s = \frac{1}{2}\Big[\underbrace{\frac{1}{2}\sum_{kl} \langle w_{kl}, e_i\rangle . e_k e_l}_{\omega^2}, e_s\Big].$$

$$= \frac{1}{4} \sum_{kl} \langle w_{kl}, e_i\rangle \left(e_k^+ e_l^+ - e_k^- e_l^-\right) e_s.$$

$$(e_s e_k) e_l = e_l^- (\widehat{e_s e_k}), \quad \text{since } e_i^- : w \longmapsto \hat{w} \lrcorner e_i + \hat{w} \wedge e_i.$$
$$= e_l^-(\hat{e}_s(e_k)).$$
$$= -e_l^-(\hat{e}_k(\hat{e}_s)).$$
$$= e_k^- e_l^-(e_s):$$

Lecture 14 (Weitzenböck formula)

$$\Rightarrow \quad D_m^2 g = \Delta_{\Delta m} g - \frac{1}{2}\sum_{ij} e_i e_j \langle \Omega, e_i \wedge e_j \rangle g .$$

$$\left(g = \sum_{S \subset \bar{n}} g_S e_S \right).$$

$$\sum_i \left(\partial_{e_i} + \frac{1}{4}\sum_{kl} \langle w_{kl}, e_i \rangle (e_k^+ e_l^+ - e_k^- e_l^-) \right)$$

$$\left(\partial_{e_i} + \frac{1}{4} \cdots \right) g .$$

$$- \sum_i \left(\partial_{\nabla_{e_i} e_i} + \frac{1}{4} \langle w_{kl}, \nabla_{e_i} e_i \rangle (e_k^+ e_l^- - e_k^- e_l^+) \right) g .$$

$$- \frac{1}{2} \sum_{ij} e_i e_j \frac{1}{4} \left[\sum_{kl} \langle R_{kl}, e_i \wedge e_j \rangle e_k e_l ; g \right].$$

this is not Ricci.
Rather $R_{kl} \in \Lambda^2$ in
Cartan formalism.

$$H^0 \longrightarrow H^1 \longrightarrow \cdots \longrightarrow H^m . \qquad \Delta^{2n} = \Delta^{4m} .$$

$$\Delta = H^0_{(q,n)} \ni \Delta^0$$

Need to only look for multiplication
by 4-vectors:

In each recursion step, we need to multiply by a 4-vector to reach $\Delta^{2m} = \Delta^{4m}$.

remainder from Δ_{ang}.

$$\mathbb{D}_M^2 \approx \cdots - \frac{1}{8} \underbrace{\sum_{ijkl} e_i e_j R_{ijkl} \left(e_k^+ e_l^+ - e_k^- e_l^-\right)}_{2S}.$$

$$\approx + \frac{1}{8} \sum_{ijkl} R_{ijkl}\, e_i^+ e_j^+ e_k^- e_l^- .$$

$$\Rightarrow \quad \mathrm{Tr}\left(H|_{\Lambda^{ev}}\right) - \mathrm{Tr}\left(H|_{\Lambda^{odd}}\right)$$

$$= 2^n \left(T_{2^n} H\right).$$

$$= \frac{2^n}{m!\, 8^m} \left(T_{2n} \sum_{ijkl} R_{ijkl}\, e_i^+ e_j^+ e_k^- e_l^-\right)^m \Bigg|_{\Delta^0}.$$

$$= \frac{1}{m!\, 2^m} \sum_{i,j,k,l,\dots,i_m,j_m,k_m,l_m} R_{ijkl} \cdots R_{i_m j_m k_m l_m} \times$$

$$\left\{ \begin{array}{l} \left(e_1^+ e_1^- \cdots + e_n^+ e_n^-\right). \\ \left(e_{i_1}^+ e_{j_1}^+ e_{k_1}^- e_{l_1}^-\right) \cdots \left(e_{i_m}^+ e_{j_m}^+ e_{k_m}^- e_{l_m}^-\right) \end{array} \right|_{\Delta^0}$$

$$(-1)^n \left(e_1^+ \cdots e_n^+\right)\left(e_1^- \cdots e_n^-\right).$$

$$\underbrace{\left(e_{i_1}^+ e_{j_1}^+ \cdots e_{i_m}^+ e_{j_m}^+\right)\left(e_{k_1}^- e_{l_1}^- \cdots e_{k_m}^- e_{l_m}^-\right)}_{\varepsilon(i,j,\dots,i_m,j_m)\ \varepsilon(k,l,\dots,k_m,l_m)}.$$

$$= \frac{\cancel{\pi_k} 2^m}{m!\,(-2)^m} \left\langle \sum_{i_1 j_1, \ldots, i_m j_m} R_{i_1 j_1} \wedge \cdots \wedge R_{i_m j_m}\, \varepsilon(i_1 j_1, \ldots, i_m, j_m),\; e_1 \wedge \cdots \wedge e_n \right\rangle.$$

Def^{n}: The Pfaffian of Reimann curvature form is:

$$\frac{1}{2^m\, m!} \sum_{i_1 j_1 \cdots i_m j_m} R_{i_1 j_1} \wedge \cdots \wedge R_{i_m j_m}\, \varepsilon(i_1 j_1, \cdots, i_m, j_m).$$

Th^m. (CGB).

M compact, $\dim M = n = 2m$ = even. Then

$$i(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) = \beta_0 \beta_1^{+} \beta_2^{-} \cdots = \chi(M).$$

$$= \left(\frac{1}{-2\pi}\right)^m \int_M \langle Pf(R), d\mu \rangle.$$

oriented measure.

Pfaffian:

- $A \in \underline{SO}(\mathbb{R}^n)$, $n = 2m$, skew symmetric matrix.

$\sim$ Lee 4 $\Rightarrow \exists!\, b \in \Lambda^2 \mathbb{R}^n$ s.t.

$$A(v) = b \llcorner v.$$

6

$\underline{\text{Def}^n}$ $\quad Pf(A) = \frac{1}{m!}\langle \underbrace{b\wedge\cdots\wedge b}_{m\text{ fold}}, e_1\wedge\cdots\wedge e_n\rangle$. $\quad \underline{n = 2m}$

$\underline{\text{Example}}$ 1. $\quad A = \begin{bmatrix} \begin{matrix} 0 & x_1 \\ -x_1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & x_2 \\ -x_2 & 0 \end{matrix} & \\ & & \ddots \end{bmatrix}$.

$$b = x_1 e_{12} + x_2 e_{34} + \cdots$$

$$Pf(A) = x_1\cdots x_m = \pm\sqrt{|\det A|}$$

$\uparrow$ only for $\underline{\text{skew } A}$.

$\underline{\text{Example}}$ 2. $\quad Pf(TAT^*)$. $\quad \underbrace{T\in\mathcal{L}(\mathbb{R}^n),\ \cancel{so}\ A\in \underline{so}(\mathbb{R}^n)}_{}$ skew.

$\overset{?}{TAT^*}$ skew.

What b correspond to TAT^*?

$$\begin{aligned}\langle TAT^*(v), u\rangle &= \langle AT^*v, T^*u\rangle \\ &= \langle b_A \llcorner (T^*v), T^*u\rangle \\ &= \langle b_A, (T^*u)\wedge T^*(v)\rangle \\ &= \langle b_A, T^*(u\wedge v)\rangle. \\ &= \langle (Tb_A)\llcorner v, u\rangle.\end{aligned}$$

$$\Rightarrow \quad b = Tb_A$$

$(Tb_A) \wedge \cdots \wedge (Tb_A) = T\mathfrak{Z}(b_A \wedge \cdots \wedge b_A)$.

and $\cancel{b_A} \cdot \langle T(b_A \wedge \cdots \wedge b_A), e_1 \wedge \cdots \wedge e_n \rangle$.

$\Leftarrow \quad = (\det T)(b_A \wedge \cdots \wedge b_A)$.

and $\underline{Pf(TAT^*) = (\det T)\, Pf(A)}$.

For skew matrix, $Pf(A)$ expanded in basis

$$Pf((A_{ij})) = \frac{1}{n!\,2^n} \sum \{\varepsilon_{i_1, j_1, \ldots, i_m, j_m}\} A_{i_1 j_1} \cdots A_{i_m j_m}.$$

makes sense with $(\mathbb{R}, \cdot, +)$ but also to $(\Lambda^{ev} M, \wedge, +)$; then this expression.

$\underbrace{\Lambda^{ev}}$ commutative!

lifts!