~~Chern-Gauss-B~~
Lecture 17.

03/11/2014.

Chern-Gauss-Bonnet Theorem:

$T: V_1 \to V_2 \quad \dim V_1, \dim V_2 < \infty$

$$\underbrace{\dim \mathcal{N}(T) - \dim(V_2/R(T))}_{i(T) := \text{ index of } T} = \underbrace{\dim V_1 - \dim V_2}_{\text{indep of } T}.$$

Motivated by this, generalise to infinite dim.

Def[n]. $T: \mathcal{H}_1 \to \mathcal{H}_2$ bdd linr. b/w Hilbert spaces.
is Fredholm if
(1) $\mathcal{R}(T)$ is closed,
(2) $\mathcal{N}(T)$, $\operatorname{coker}(T) = \mathcal{H}_2/\mathcal{R}(T)$ are finite dim.

Ex. M cpct, Riem mfld. Say $D_M := d_M + \delta_M$.
as operator fr $H^1(M, \Lambda M) \to L_2(M, \Lambda M)$.
is Fredholm

Problem. Relate $i(D_M)$ to geometry and topology of M.

Note: ~~~~ $i(\mathbb{D}_M) = 0$ always, because

dim ker $(\mathbb{D}_M)$ = dim coker $(\mathbb{D}_M)$. ~~~~

because $\mathbb{D}_M^* = -\mathbb{D}_M$.

Instead, consider $\mathbb{D}_M : L_2(M, \Lambda^{ev} M) \to L_2(M, \Lambda^{od} M)$.

Then,

$$i(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) = \dim(N(-)) - \dim R(-)^{\perp}$$

$$= \dim N(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) - \dim N(\mathbb{D}_M : \Lambda^{od} \to \Lambda^{ev})$$

Topology $\to$ $[= (\beta_0 + \beta_2 + \cdots +) - (\beta_1 + \beta_3 + \cdots)$

$$=: \chi(M). \longleftarrow \text{Euler Characteristic.}$$

Note: $*: L_2(M, \Lambda^{ev}) \to L_2(M : \Lambda^{od})$ and

So, $\beta_k = \beta_{n-k}$ and hence, if

$\dim M = n =$ odd, then $i(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) = 0$.

So, only consider $\underline{\dim M = n = \text{even} = 2m}$.

We use the heat equation method. (Gilkey's book).

$$i(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) = \dim N(\mathbb{D}_M|_{\Lambda^{ev}}) - \dim N(\mathbb{D}_M|_{\Lambda^{od}}).$$

$$= \dim \{ N(\mathbb{D}_M^2|_{\Lambda^{ev}}) - \dim N(\mathbb{D}_M^2|_{\Lambda^{od}}).$$

$\mathbb{D}_M$ is normal

(*) Heat eqⁿ method: $= \operatorname{Tr}(e^{t\mathbb{D}_M^2|_{ev}}) - \operatorname{Tr}(e^{t\mathbb{D}_M^2|_{od}}).$ $\quad \underline{\forall t > 0.}$

can take limit

$\lim_{t \to 0^+}$ ______ to obtain <u>local</u> operator.

<u>Key observation:</u> A appropriate operator, then non-zero eigenvalues of A^*A and AA^* are the same.

$$\operatorname{Tr}(e^{t\mathbb{D}_M^2|_{ev}}) = \overbrace{(1 + \cdots + 1)}^{\text{zero eigenvalue}} + (e^{-t\lambda_1^2} + \cdots + e^{-t\lambda_p^2} + \cdots)$$

$$\operatorname{Tr}(e^{t\mathbb{D}_M^2|_{od}}) \; \underbrace{(1 + \cdots + 1)}_{\text{zero eigenvalue}} + (e^{-t\varsigma_1^2} + \cdots + e^{-t\varsigma_p^2} + \cdots)$$

<u>Claim:</u> $\lambda_i = \varsigma_i^2$, the infinite sum converges, and so (*) holds!

Want to compute $f_t := e^{t\mathbb{D}_M^2} f$, is the sol^n to the heat equation.

$$\begin{cases} \partial_t f_t = \mathbb{D}_M^2 f_t & t > 0. \\ f_t\big|_{t=0} = f. \end{cases}$$

Model example: $\mathbb{D}_M^2 :=$ standard Laplace operator in $\mathbb{R}^n$.

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

for $f : \mathbb{R}^n \to \mathbb{R}$.

Ansatz for $e^{t\mathbb{D}_M^2}$:

$$\underset{f \in L_2(M;\Lambda M)}{H_t f(p)} = \frac{1}{(4\pi t)^{n/2}} \int_M e^{-\frac{d(p,q)}{4t}} \sum_{k=0}^{m} t^k H^k(p,q) f(q)\, dq.$$

$= n/2 = \frac{\dim M}{2}$

$d(p,q) =$ geodesic distance b/w p, q.

$H^k(p,q) : \mathcal{L}(\Lambda^q M, \Lambda^p M)$.

Remark: ① The integral is only over the fibre p.

② We need information about $H^k(p,q)$.

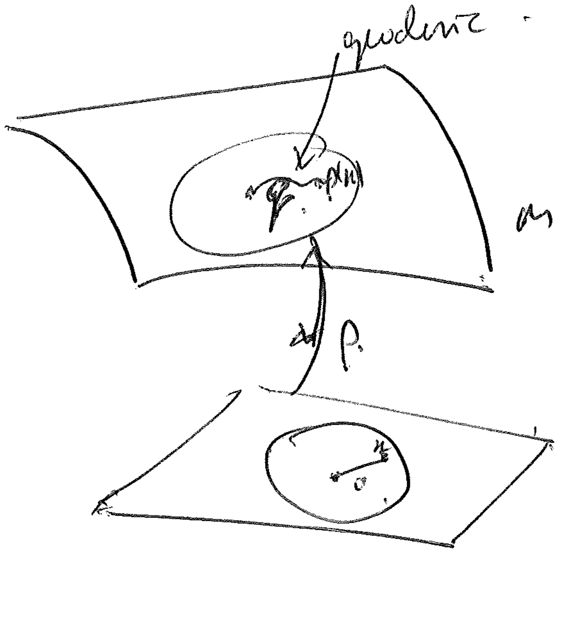
Lemma 12.14.

$$\left(\partial_t - \mathbb{D}_M^2\right) H_t f(p) = \int_M \frac{1}{(4\pi t)^m} e^{-\frac{d^2(p,q)^3}{4t}} \Bigg(\frac{1}{t^{m+1}} \left(\nabla_{r_q} + \frac{1}{4}\partial_{r_q}(\ln g_p)\right) H^0(p,q)$$

$(\det g_{ij})_p$

$$+ \sum_{k=1}^{m} \frac{1}{t^{m+1-k}} \left(\nabla_{r_q} + \tfrac{1}{4}\partial_{r_q}\ln g + k\right) H^k(p,q) \cancel{+ \sum^{m} \frac{1}{t^{m+1+k}} \left((\nabla_{r_q} + \frac{1}{4} \partial_{r_q} \ln g)\right) H^k(p,q)} - D_M^2 H^{k-1}(p,q) \Bigg)$$

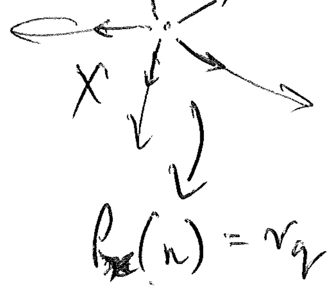
$$- \mathbb{D}_M^2 H^m(p,q) \Bigg) f(q)\, dq.$$



Compute in normal coordinates! ~~Normal~~

$\ln g = \ln(\det(g_{ij}(x)))$. $x \in$ nbh of q.

r_q = ~~vector~~ radial vector field X pushforward by $\exp$.



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Want to find $H^k(p,q)$ by asking:

$$\begin{cases} \left(\nabla_{r_q} + \frac{1}{4}\partial_{r_q}(\ln g)\right) H^0(p,q) = 0 & \text{(A)} \\ \left(\nabla_{r_q} + \frac{1}{4}\partial_{r_q}(\ln g) + k\right) H^k(p,q) - D_q H^{k-1}(p,q) = 0. & \text{(B)} \end{cases}$$

<u>Define</u> $\tilde{H}^k(p,q)$, for q close to p.

(I) $p=q$: $\tilde{H}^0(q,q) = I$. (because we want $H_t f \to f$ as $t \to 0$)

(II) $p \neq q$: $\tilde{H}^0(p,q)$ solves (A), which is an ODE since we're simply considering geodesic r_q.

(III). Higher k, ie $k \geq 1$: Want $\tilde{H}^k(q,q)$ determined by $\tilde{H}^{k-1}$, b/c

$$\nabla_{r_q} + \frac{1}{4}\partial_{r_q}(\ln g) = 0 \quad \text{in}$$

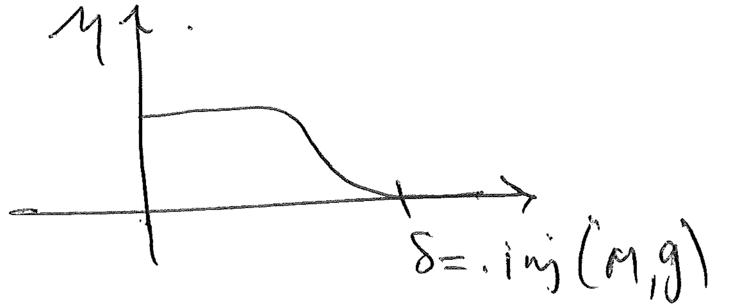
need conditions.

~~for $p \neq q$:~~

For p and q far away,

$$H^k(p,q) := \eta(d(p,q))\,\tilde{H}^k(p,q).$$

where



<u>Now, compute:</u>

$$(\partial_t - \mathbb{D}_M^2)\left(H_t f - e^{t\mathbb{D}_M^2} f\right)$$

$$= -\frac{1}{(4\pi)^m} \int_M e^{-\frac{d^2(p,q)}{4t}} H^m(p,q) f(q)\, dq.$$

no t dependence!

$$k_t(p,q) = -\frac{1}{(4\pi)^m} e^{-\frac{d^2(p,q)}{4t}} H^m(p,q) \cancel{f(q)}.$$

(¢) <u>Claim</u> $\sup_{\substack{p,q\in M\\ t>0}} k_t(p,q) < \infty$ along w. some smoothness!

Integrate in t:

$$H_t f - e^{t\mathbb{D}_M^2} f = \int_0^t e^{(t-s)\mathbb{D}_M^2} k_s f\, ds \quad \cancel{k_s f\, ds}.$$

$$k_s f(p) = \int_M k_t(p,q) f(q)\, dq.$$

$$\| H_t - e^{-t\mathbb{D}_M^2} \|_{Tr} \leq \int_0^t \| \cancel{B_s} e^{(t-s)\mathbb{D}_M^2} k_s \|_{Tr}.$$

(Trace norm $\|\cdot\|_{Tr} = \sum_j \lambda_j$ of singular values, and is stronger!)

$$\leq \int_0^t \underbrace{\|e^{(t-s)\mathbb{D}_M^2}\|_{L^2\to L^2}}_{=1} \underbrace{\|k_s\|_{Tr}}_{\leq C \text{ (By } \phi)} ds.$$

$$\leq Ct \to 0 \text{ as } t \to 0.$$

<u>Go back to:</u>

$$i(\mathbb{D}_M : \Lambda^{ev} \to \Lambda^{od}) = \lim_{t\to 0^+} (Tr(e^{t\mathbb{D}_M^2}|_{\Lambda^{ev}}) - Tr(e^{t\mathbb{D}_M^2}|_{\Lambda^{od}}))$$

$$= \lim_{t\to 0^+} (Tr(H_t|_{\Lambda^{ev}}) - Tr(H_t|_{\Lambda^{od}})).$$

But H_t integral op., so, Tr is diag of kernel:

$$= \lim_{t\to 0^+} \int_M Tr(H_t(q,q)|_{\Lambda^{ev}}) - Tr(H_t(q,q)|_{\Lambda^{od}}) \, dq.$$

$H_t(q,q)$ are matrices over each fibre q!

$$= \lim_{t \to 0^+} \frac{1}{(4\pi)^m} \sum_{k=0}^{m} \frac{1}{t^{m-k}} \int_M \left(\mathrm{Tr}\left(H^k(q,q)\right|_{ev}\right) - \mathrm{Tr}\left(H^k(q,q)|_{od}\right) dq.$$

<u>Claim</u>. only term that can survive is $k = m$, because we <u>have</u> verified that the operators are trace class, and hence, if a $m \neq k$ term survives, then this limit would blow up. This contradicts that $H_t - e^{t\mathbb{D}_M^2}$ is trace class.

$$= \frac{\cancel{2}\;1}{(4\pi)^m} \int_M \mathrm{Tr}\left(H^m(q,q)\right|_{\Lambda ev}) - \mathrm{Tr}\left(H^m(q,q)\right|_{\Lambda od}) dq$$

Remains to compute $\mathrm{Tr}\, H^m(q,q)$ restricted to Λ^{od} and Λ^{ev}. We don't want to define and solve for even H^{k+}, $k \leq m$, because we only need trace information.

Go down to $\mathcal{R}(\Lambda\mathbb{R}^n) \cong \Delta\mathbb{R}^{2n} \leftarrow$ inner product with sign $= 0$.

$\{e_S' e_T\}_{S \subset \bar{n},\, \$ \subset \bar{n}}$, a good basis to compute.