Lecture 16 30/10/2014.

Topology for Hodge decompositions.

M = cpct Reim mfld. (no bdy).

$$L^2(M;\Lambda M) = \mathcal{R}(d_M) \oplus \underbrace{\big(N(d_M) \cap N(\delta_M)\big)}_{=: \mathcal{H}(M).\ \text{finite dim.}} \oplus \mathcal{R}(\delta_M).$$

<u>Def</u>n. (de Rham) ~~the~~ cohomology spaces for M are

$$\mathcal{H}(M) = \bigoplus_{k=1}^{n} \mathcal{H}^k(M).$$

$$\mathcal{H}^k(M) = \{ F \in L^2(M;\Lambda^k M) : d_M F = 0,\ \delta_M F = 0 \}.$$

Betti numbers $\beta_k(M) := \dim \mathcal{H}^k(M)$.

<u>Remark</u> $\beta_k(M)$ are topological. The variation in metric does not alter $\beta_k(M)$, even though $\mathcal{H}(M)$ will change.

<u>Prop</u> If $\rho : M_1 \longrightarrow M_2$ is a diffeomorphism, then $\beta_k(M_1) = \beta_k(M_2)$ $\forall k$.

Pf. $N(\delta_M) = \mathcal{R}(d_M)^{\perp}$, so

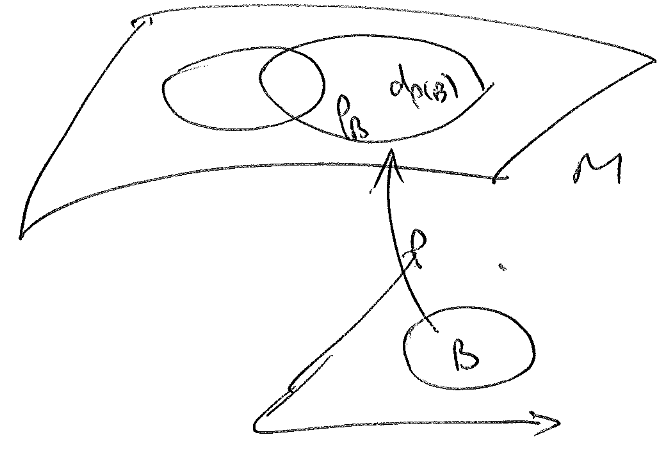
$$\mathcal{H}(M) \cong N(d_M)/\mathcal{R}(d_M).$$

Pull back gives isomorphisms.

$$\frac{N(d_{M_1})\cancel{/R(d_{M_1})}}{R(d_{M_1})} \quad \begin{matrix} \xleftrightarrow[\rho^*]{\cong} \\[1em] \xleftarrow[\rho^*]{\cong} \end{matrix} \quad \frac{N(d_{M_2})}{R(d_{M_2})}.$$

Moreover, $\mathcal{H}^k(M_1) \xleftarrow[\cong]{\rho^*} \mathcal{H}^k(M_2)$, by same argument and restriction to k-forms (Note: we need to quotient by $(k-1)$-form once under d since $N^k(\delta_M) = \mathcal{R}^{k-1}(d_M)$. This is the topological feel in the sense of algebraic top).

Poincaré Theorem:



Consider charts $\rho(B) \subset M$ with $B =$ unit ball in $\mathbb{R}^n$, and that the ext. derivative

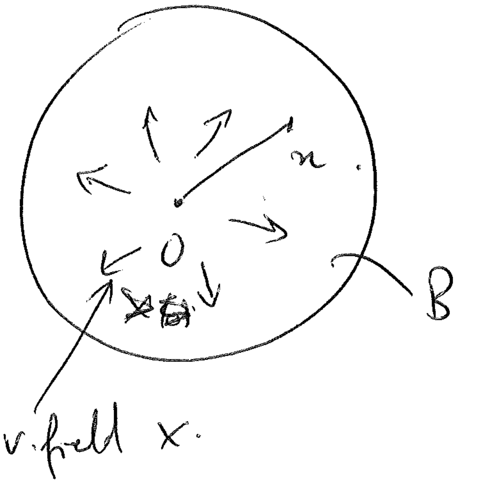
$$d_{\rho(B)} = \rho^{*-1} d_B \rho^* \quad \text{as an operator in } L^2(\rho(B); \Lambda B).$$

Th^m. If $\nabla \wedge F = 0$ in $B \subset \mathbb{R}^n$ for $F \in C^1(B, \Lambda^k \mathbb{R}^n)$ with $k \geq 1$. Then, $F = \nabla \wedge u$ where

$$u(x) := X(x) \lrcorner \int_0^1 F(tx)\, t^{k-1}\, dt.$$

where $X(x) = \sum_i x_i e_i$.

(Cheating a little b/c we have a boundary in B.)



Pf.

$$\nabla \wedge u(x) = \sum_i e_i \wedge \left(e_i \lrcorner \int_0^1 F(tx)\, t^{k-1}\, dt + X \lrcorner \int_0^1 \partial_{x_i} F(tx)\, t^{k-1}\, dt \right).$$

$$= \sum_i e_i \wedge \left(e_i \lrcorner \int_0^1 F(tx)\, t^{k-1}\, dt + X \lrcorner \int_0^1 (\partial_{x_i} F)(tx)\, t^{k}\, dt \right).$$

$$= k \int_0^1 F(tx)\, t^{k-1}\, dt +$$

$$x_i \quad \sum_i \langle X, e_i \rangle \int_0^1 (\partial_{x_i} F)(tx)\, t^{k}\, dt.$$

$$- \sum_i x \lrcorner \underbrace{\cancel{\left(e_i \wedge \int_0^1 (\partial_{x_i} F)(tx)\, t^{k}\, dt\right)}}_{\nabla \wedge F = 0} \; 0 .$$

$$x_i \int_0^1 (\partial_{x_i} F)(tx)\, t^{k}\, dt = \int_0^1 \frac{d}{dt} F(tx)\, t^{k}\, dt$$

So, $\nabla \wedge u(x) = \int_0^1 \frac{d}{dt} F(tx) \, t^k \, dt + k \int_0^1 F(tx) t^{k-1} dt$

$$= F(x) - 0 . \qquad \boxtimes$$

<u>Note:</u> • By varying the base point $p = 0$ and averaging, we get an L^2-bdd map $F \mapsto u$. (§8.7).

• $\beta_k(B) = 0 \quad k = 1, 2, \dots, n$.

$\beta_0(B) = 1$. clearly. ($H^0(M) = N(d_M) / \cancel{R(d_M)} = N(d_M)$. $= \{\text{consts}\}$).

Čech Cohomology.

<u>Aim:</u> A finite algorithm for calculating $\beta_k(M)$. for a given manifold M.

<u>Problem:</u> $\dim H^k(M) = \dim \dfrac{N(d_M|_{\Lambda^k})}{R(d_M|_{\Lambda^{k-1}})}$? ⟵ infinite dim!!!

<u>Algebraic machinery</u>

• A finite covering of M by open subsets.

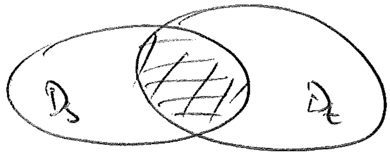
$$M = D_1 \cup \dots \cup D_N .$$

• A k-fold intersection. $D_S := D_{s_1} \cap \dots \cap D_{s_k}$. indexed by $S \subset \bar{N} = \{1, \dots, N\}$.

• (Pre-) sheafs.

$F(D)$ = collection of linear spaces, one for each D_S, $S \subset \overline{N}$

Condition:



with "restriction maps"!!

Ex: $F(D) = \mathbb{R}$. (for each intersection, $\exists$ a $\mathbb{R}$-number)

$F(D) = N(d) \subset L^2$, for D_S, L^2 multivector fields that are closed.

$F(D) = D(d)$.

$F(D) = \mathbb{Z}_2 = \{\pm 1\}$ (multiplicatively) $\leftarrow$ needed for Spinors.

~~Ch~~ $\check{C}$ech ~~k~~-cochains: f is a collection of vectors one in each $F(D_S)$ for each D_S with $|S| = k+1$.

multivector notation: $\langle f, e_S \rangle \in F(D_S) :=$ value of f at D_S.

$$\partial_K : \underbrace{C^k(\underline{D}, F)}_{\{k\text{-cochains}\}} \longrightarrow C^{k+1}(\underline{D}, F).$$

$$\underset{|S| = k+2}{\langle \partial_k f, e_S \rangle} := \sum_{i=1}^{N} \langle f, \underset{\left\{ \begin{array}{l} e_{S \setminus \{i\}} \quad i \in S \\ 0 \end{array} \right.}{e_i \lrcorner e_S} \rangle$$

assume linearity so only possible.

Chasing "Let's check it for Čech" - Andreas.

Claim: $\partial_{k+1} \circ \partial_k = 0$.

$$\langle \partial_{k+1} \partial_k f, e_S \rangle = \sum_i \langle \partial_k f, e_i \lrcorner e_S \rangle \big|_{D_S}.$$

$$= \sum_i \Big(\sum_j \langle f, \underbrace{e_j \lrcorner (e_i \lrcorner e_S)}_{\underbrace{e_i \wedge e_j}_{\text{alternating}} \lrcorner e_S} \rangle \big|_{D_{S \setminus \{i,j\}}} \Big) \Big|_{D_S}.$$

$$= 0.$$

Def^{n} Čech cohomology spaces.

$$\mathcal{H}^k(\underline{D}, F) := N(\partial_k) / R(\partial_{k+1}).$$

Th^{m}. $M = \bigcup_{i=1}^N D_i$, cpct, and assume the cover $\{D_i\}$ is good - i.e., all intersections are diffeomorphic to balls. Then

$$\beta_k(M) = \underbrace{\dim(\mathcal{H}^k(M))}_{\text{indep of the good cover.}} = \dim \underbrace{\mathcal{H}^k(\underline{D}, \mathbb{R})}_{\text{Indep of } d_M, \delta_M.}$$

Note:
Algorithm is 2nd equality, and $N(\partial_k)$, $R(\partial_{k-1})$ are finite dim for a good cover and since the sheaf is $\mathbb{R}$. Ie, finitely computable!

A main ingredient in the proof: a Poincaré type thm.

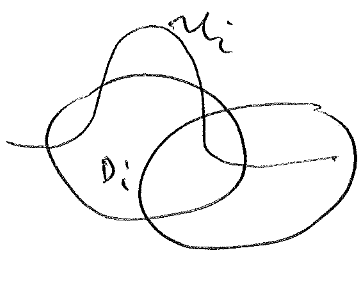
<u>Lemma</u>. Consider the sheaf

$$F(D_s) = \mathcal{D}(d_{D_s}; \Lambda^k) = \{ f \in L^2(D_s; \Lambda^k M) : d_M f \in L_2(D_s; \Lambda^{k+1} M)\}.$$

Then, $H^k(\underline{D}, F) = 0$ if $k \geq 1$.

<u>Pf.</u> Assume $f \in C^k(\underline{D}, F)$ and $\partial_k f = 0$.

Take p.o.u. $\{\eta_i\}_{i=1}^N$ supp $\eta_i \subset D_i$



Define $g \in C^{k-1}(\underline{D}, F)$ by.

$$\underbrace{\langle g, e_t\rangle}_{\substack{|t|=k \\ F(D_t)}} := \underbrace{\sum_i \eta_i \langle f, e_i \wedge e_t\rangle}_{F(D_{t\cup\{i\}})} . \qquad D_t \cap D_i \subset D_t$$

(extend each $\eta_i \langle f, e_i\wedge e_t\rangle$ by 0 to whole of D_t)

Need: $\partial_{k-1} g = f$

$$\langle \partial_{k-1} g, e_s\rangle = \sum_j \langle g, e_j \lrcorner e_s\rangle \Big|_{D_s}$$

$$= \sum_j \Big(\sum_i \eta_i \langle f, e_i \wedge \underbrace{(e_j \lrcorner e_s)}_{\delta_{ij} e_s - e_j \lrcorner (e_i \wedge e_s)} \rangle \Big)\Big|_{D_s}$$

$\uparrow$ o.n. frame.

$$= \sum_i \eta_i \langle f, e_s\rangle\Big|_{D_s} - \Big(\sum_i \eta_i \Big(\sum_j \langle f, e_j \lrcorner (e_i \wedge e_s)\rangle\Big)\Big)\Big|_{D_s}$$

$\partial_u f = 0 \Rightarrow \quad \sum_i \langle f, e_i \lrcorner (e_i \wedge e_s) \rangle = \langle \partial_u f, e_s \rangle = 0$ ∎

<u>Note</u> $\mathcal{H}^k(\underline{D}, F) = \{0\} \quad k \geq 1,$

But $\mathcal{H}^k(\underline{D}, \mathbb{R}) \neq \{0\}$ since it is the Betti numbers. So, what is different?

We need η_i to ~~ext~~ cut off and extend $y \in C^{k-1}(\underline{D}, F)$ to all of D_S by 0. But if this was a $\neq 0$ real number, we <u>cannot</u> do this.

$F(D_S)$ — "a fine sheaf", i.e., invariant under smooth cutoffs.

<u>Future</u>. β_k are the k-dim top. obstructions.