Lecture 15. 27/10/2014.

Hodge decompositions of multivector fields.

Setup: Manifold $\mathcal{M}$, compact (essential).

Reimannian.

ΛM - exterior bundle over $\mathcal{M}$.

d_M, δ_M - exterior and interior derivative on ΛM.

~ Regard them as closed, densely-defined operators in $L^2(M, \Lambda M)$.

- operators are Nilpotent. $d_M^2 = \delta_M^2 = 0$.
I.e. $\mathcal{R}(d_M) \subset \mathcal{N}(d_M)$, $\mathcal{R}(\delta_M) \subset \mathcal{N}(\delta_M)$.

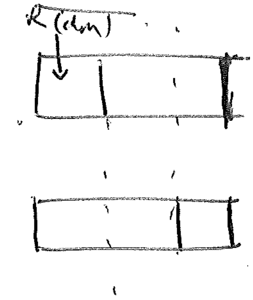
Remark. For a general $L : \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$,
$\mathcal{R}(L)$ is not closed, but $\mathcal{N}(\delta_M)$ is.

Abstract duality result: T closed in $\mathcal{H}$,
Then, $\mathcal{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T^*)}$.

Apply this to $\mathcal{H} = L^2(\mathcal{M}; \Lambda M)$ with $T = d_M$
or δ_M.

$$L_2(M;\Lambda M) = \overline{\mathcal{R}(d_M)} \oplus N(\delta_M) \qquad (T = \delta_M)$$

$$L_2(M;\Lambda M) = N(d_M) \oplus \overline{\mathcal{R}(\delta_M)} \qquad (T = d_M)$$



by Nilpotency.

$$\Rightarrow \quad L_2(M;\Lambda M) = \overline{\mathcal{R}(d_M)} \oplus \big(N(d_M) \cap N(\delta_M)\big) \oplus \overline{\mathcal{R}(\delta_M)}.$$

Think of this as a pre-Hodge decomposition.

Thm. If M is compact, then $\mathcal{R}(d_M)$, $\mathcal{R}(\delta_M)$ are closed, and

$$\mathcal{H}(M) := N(\delta_M) \cap N(d_M) \qquad \text{(cohomology space)}$$

is finite dimensional.

Note: there is an analogous Hodge splitting for bounded Euclidean domains, see §8. Advantage: flat, hold $\Omega \subset \mathbb{R}^n$. Negative: boundary conditions need to be imposed.

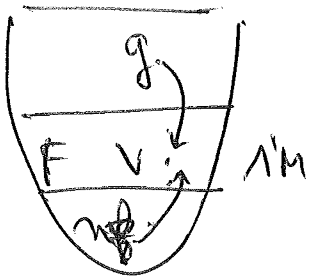
Example: A vectorfield $F \in L^2(M;\Lambda M)$ splits

$$F = \nabla_M u + v + \delta_M g$$

$\nabla_M u$: function; v: v.field; $\delta_M g$: bivector.

$$\begin{cases} \operatorname{curl}_M v = 0 \\ \operatorname{div}_M v = 0 \end{cases}$$

$\boxed{\begin{array}{l} \operatorname{curl}_M = d_M \text{ on v.field} \\ \operatorname{div}_M = \delta_M \text{ on v.field} \end{array}}$



This is called the Helmholtz decomposition.

$\nabla_M u + v$ — curl free.

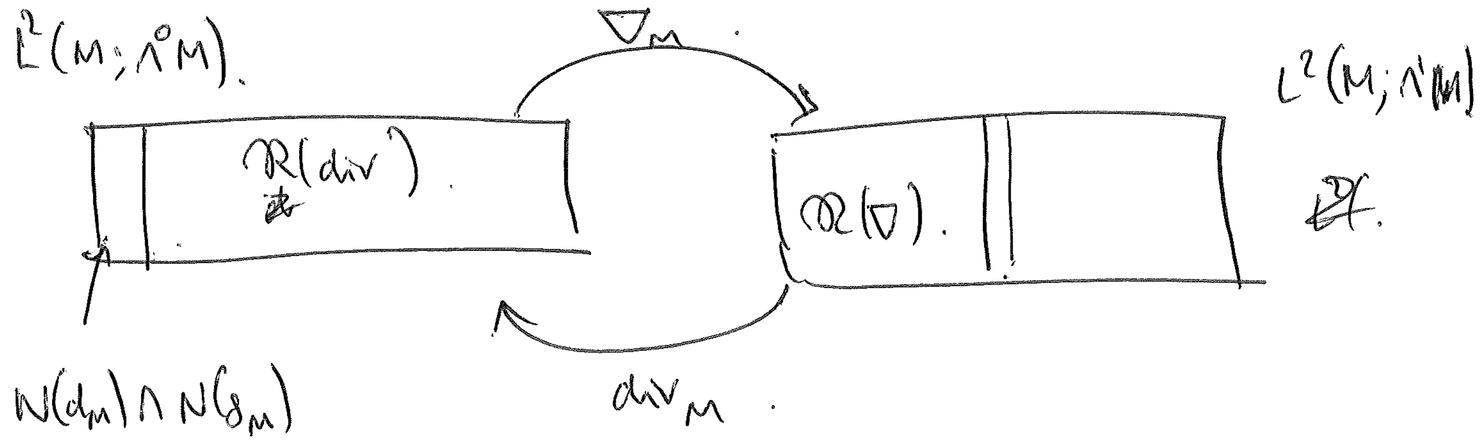
$v + \delta_M q$ — div free.

<u>Example</u>. Use Hodge-decomposition to solve the Poisson equation.

$$\Delta_M u = f, \qquad f \text{ given scalar function.}$$

$$f \in L^2(M).$$

$$\underbrace{\delta_M d_M u}_{\text{since we're on scalars.}} = f.$$



$N(d_M)\cap N(\delta_M)$
$=$ constants

Since $\delta_M u = 0 \ \forall u \in L^2(M;\Lambda^0 M)$.

Solve $\Delta_M u = f \Leftrightarrow \mathrm{div}_M \nabla_M u = f$.

But $\mathrm{div}_M v = f$ iff f orthogonal to $N(d_M)\cap N(\delta_M)$
$=$ constants.

So,

Ie, we need the moment condition that

$$\int_M f \, d\rho = 0.$$

Obtain $V \in \mathcal{R}(\nabla)$ s.t. $\operatorname{div} v = f$. In fact, v is unique.

But $v \in \mathcal{R}(\nabla) \Rightarrow \exists\, u \in \mathcal{R}(\operatorname{div})$ s.t.

$$v = \nabla_M u.$$

This is unique because ~~it is~~ $u + N(d_M) \cap N(\delta_M)$.

<u>Note:</u> The fact that $\mathcal{R}(\nabla)$, $\mathcal{R}(\operatorname{div})$ being closed is <u>crucial</u>, because we won't have $\int_M f \, d\rho = 0 \to \exists!\, v \in \mathcal{R}(\nabla)$ s.t.

$$\operatorname{div} v = f.$$

We would have $f \in \overline{\mathcal{R}(\operatorname{div})}$, which means we have a sequence $v_j \in \mathcal{R}(\nabla)$ s.t.

$$\operatorname{div} v_j \to f.$$

<u>Pf of Hodge-decomposition:</u>

<u>Prop</u> (Gaffney inequality). $\forall F \in C^\infty(M; \Lambda F)$,

$$\int_M |\nabla \otimes F|^2 \, d\rho \leq \int_M \left(|d_M F|^2 + |\delta_M F|^2\right) d\rho + c \int |F|^2 \, d\rho.$$

Pf of Hodge decomposition Thm:

Consider Hodge Dirac operator on M,

$$\mathbb{D}_M = d_M + \delta_M :$$

Consider $\mathbb{D}_M : \cancel{\#}W^{1,2}(M;\Lambda M) \to L^2(M,\Lambda M)$,

<u>bdd</u> operator, where $\|F\|^2_{W^{1,2}} := \|\nabla \otimes F\|^2_{L^2} + \|F\|^2_{L^2}$.

Clearly, we have that

$$\|\mathbb{D}_M F\|_{L^2} \lesssim \|F\|_{W^{1,2}} \lesssim \|\mathbb{D}_M F\|_{L^2} + \|F\|_{L^2}.$$

Gaffney inequality.

But $\mathcal{R}(d_M) \perp \mathcal{R}(\delta_M)$,

$$\|\mathbb{D}_M F\|^2_{L^2} = \|d_M F\|^2_{L^2} + \|\delta_M F\|^2_{L^2}.$$

$\Rightarrow$ ~~$\mathbb{D}_M$~~ $\mathbb{D}_M$ Fredholm operator

$\Leftrightarrow$ $\begin{cases} N(\mathbb{D}_M), \text{ finite dim., and also co-kernel.} \\ \mathcal{R}(\mathbb{D}_M) \text{ closed.} \end{cases}$

Need: Rellich compactness thm!

$$W^{1,2}(M) \hookrightarrow L^2(M).$$

compact. $\Leftarrow$ unit ball mapped to pre cpct set.

But $\mathcal{R}(\mathbb{D}_M) = \mathcal{R}(d_M) \oplus \mathcal{R}(\delta_M)$.

$\Rightarrow$ $\mathcal{R}(d_M)$, $\mathcal{R}(\delta_M)$ is closed.

$N(\mathbb{D}_M) = N(d_M) \cap N(\delta_M)$ — by def^{n}.

<u>Proof of prop:</u>

$$\int_M \left(|d_M F|^2 + |\delta_M F|^2 \right) = -\int_M \langle (\delta_M d_M + d_M \delta_M) F, F \rangle$$
$$= -\int_M \langle \mathbb{D}_M^2 F, F \rangle.$$

~~Compute~~ Define $v(p) = \sum_{i=1}^{n} \langle \nabla_{\bar{e}_i} F, F \rangle \bar{e}_i$

$$\int_M \operatorname{div} v = \sum_{i=1}^{n} \langle \bar{e}_i, \underbrace{\nabla_{\bar{e}_i} \left(\sum_j \langle \nabla_{\bar{e}_j} F, F \rangle \bar{e}_j \right)}_{\sum_j \partial_{\bar{e}_i} \langle \nabla_{\bar{e}_j} F, F \rangle \bar{e}_j + \langle \nabla_{\bar{e}_j} F, F \rangle \nabla_{\bar{e}_i} \bar{e}_j} \rangle.$$

$$\underbrace{\partial_{\bar{e}_i} \langle \nabla_{\bar{e}_j} F, F \rangle}_{\langle \nabla_{\bar{e}_i} \nabla_{\bar{e}_j} F, F \rangle + \langle \nabla_{\bar{e}_j} F, \nabla_{\bar{e}_i} F \rangle}$$

$$= \int_M \sum_i \langle \nabla_{\bar{e}_i} F, \nabla_{\bar{e}_i} F \rangle + \langle \nabla_{\bar{e}_i} \nabla_{\bar{e}_i} F, F \rangle$$
$$+ \sum_{i,j} \langle \nabla_{\bar{e}_j} F, F \rangle \underbrace{\langle \nabla_{\bar{e}_i} \bar{e}_j, \bar{e}_i \rangle}_{\text{by o.n.} \; = -\langle \bar{e}_j, \nabla_{\bar{e}_i} \bar{e}_i \rangle}$$

and

$$\langle \nabla_{\bar{e}_i}\nabla_{\bar{e}_i}F, F\rangle + \sum_{ij} \underline{\qquad\qquad\qquad}$$

$$= \int_M \langle \underbrace{\textstyle\sum_i \nabla_{\bar{e}_i}\nabla_{\bar{e}_i}F - \nabla_{\nabla_{\bar{e}_i}\bar{e}_i}F}_{\Delta F \;-\; \text{Lap. Beltrami}}, F\rangle$$

(Really, illegal to have $\int$ since not global frame. Rather, ... $\operatorname{div} v = \langle \Delta F, F\rangle$ pointwise.)

Need $D_M^2 - \Delta_{\wedge M}$ hold!

$$D_M^2 F = \sum_{ij} \bar{e}_i \nabla_{\bar{e}_i}(\bar{e}_j \wedge \nabla_{\bar{e}_j} F).$$

$$= \sum_{ij} \bar{e}_i \Big((\nabla_{\bar{e}_i}\bar{e}_j) \nabla_{\bar{e}_j}F$$

$$+ e_j \nabla_{\bar{e}_i}\nabla_{\bar{e}_j} F \Big).$$

$$= \sum_i \nabla_{\bar{e}_i}\nabla_{\bar{e}_i}F + \sum_{i<j} \big(\underline{\qquad\qquad} \big).$$

$$\vdots$$

$$= \Delta F + \boxed{\begin{aligned}&\textstyle\sum_i \nabla_{\nabla_{\bar{e}_i}\bar{e}_i}F \\ &+ \textstyle\sum_{ij} \bar{e}_i\bar{e}_j \big(\nabla_{[\bar{e}_i,\bar{e}_j]}F - \langle \Omega_M, \hat{e}_i\wedge\bar{e}_j\rangle F \\ &+ \textstyle\sum_{ij} \bar{e}_i \nabla_{\bar{e}_i}e_j \nabla_{\bar{e}_j} F.\end{aligned}} \quad (= 0!\ \text{Claim!})$$

$$= \sum_i \nabla_{\nabla_{\bar e_i} e_i} F$$

$$+ \sum_{ij} \bar e_i \bar e_j \nabla_{\nabla_{\bar e_i} \bar e_j} F$$

Exercise!

$$+ \sum_{ij} \Big(\underbrace{\langle e_i, \nabla_{\bar e_i} \bar e_j \rangle}_{-\langle \nabla_{\bar e_i} \bar e_i, \bar e_j \rangle} + e_i \wedge \nabla_{\bar e_i} \bar e_j \Big) F$$

$= 0$

$\boxtimes$