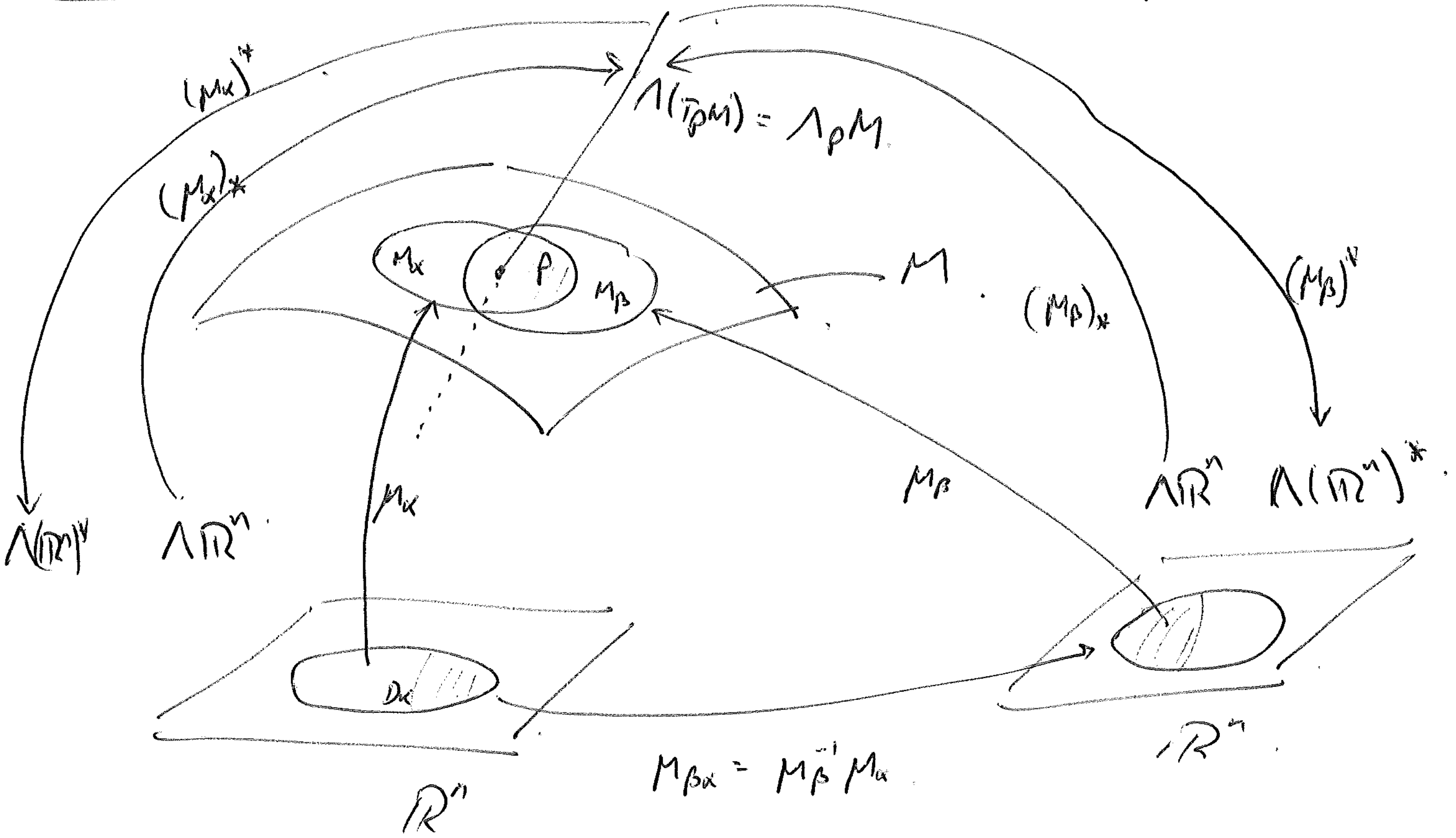
Lecture 14 24/10/2014



We care about $\Lambda(\mathbb{R}^n)^*$ and $\Lambda\mathbb{R}^n$ separately because:

Def^n. $F \in C^\infty(M, \Lambda M)$, then the exterior derivative:

$$dF(x) := (\mu_\alpha^*)^{-1} d_{\mathbb{R}^n}(\mu_\alpha^* F)(x) \quad , \text{ for } x \in M_\alpha.$$

$$\delta F(x) := (\tilde{\mu}_\alpha)_* \, \delta_{\mathbb{R}^n} (\tilde{\mu}_\alpha)_*^{-1} F(x), \quad x \in M_\alpha.$$

↑ normalized manifold.

Lecture 8/9 $\Rightarrow$ d, δ are indep of α.

Throughout, we use metric to identify $\Lambda M \cong \Lambda^* M$.

Prop Let $\{\bar{e}_i\}$ be a frame for TM, and let ∇ denote the Levi-Civita covariant derivative on TM. Then,

$$d_M F = \sum \bar{e}_i^* \wedge \nabla_{\bar{e}_i} F.$$

$$\delta_M F = \sum_i \bar{e}_i^* \lrcorner \nabla_{\bar{e}_i} F.$$

<u>Pf</u> (I) $\Rightarrow$ (II) By Hodge $*$-duality.

To prove (I) w.l.o.g., assume $\bar{e}_i := (\mu_\alpha)_* e_i$. Can be assume to be a coordinate frame, and F = scalar or v-field is sufficient. ~~to~~ be fruit, it is enough to write $F = \bar{e}_k^*$. in the vector case, the scalar case is immediate!.

<u>Need</u>: $\sum_i \bar{e}_i^* \wedge \nabla_{\bar{e}_i} \bar{e}_k^* = 0$, by explicit computation.

Express now d_M, δ_M via ∇ on $\wedge M$ and now ∇ on TM.

Prop. Fix a ON-frame $\{\bar{e}_i\}$ for TM.
and consider L.C. cov. derivative on ΛM.

If $F = \sum f_s \bar{e}_s$ is a multivector field on M,

then,
$$\nabla_u F = \sum_s \partial_u f_s \cdot \bar{e}_s + f_s \cdot \underbrace{\tfrac{1}{2}[\omega^2(u), \bar{e}_s]_\Delta}_{\nabla_u \bar{e}_s}$$

Δ ← diff commutator.

where $[a,b]_\Delta = a \Delta b - b \Delta a$.
and $\omega^2(u) := \sum_{i<j} \langle w_{ij}, u\rangle \, \bar{e}_i \wedge \bar{e}_j$.
w_{ij} - Christoffel symbols for (TM, ∇).

ω^2 ← refers to bivector object.

Note: ω^2 depends on the frame, just like Christoffel symbols.

Pf. Need $\nabla_u \bar{e}_s = \frac{1}{2}[\omega^2(u), \bar{e}_s]$.
W.l.o.g., assume $\bar{e}_s = \bar{e}_i$, ie basis vectorfield.
Exercise.

Curvature according to Cartan. for TM and ΛM.

Consider vector bundle E over M with a covariant derivative ∇

Fix a frame $\{\bar{e}_i\}$ for E

$$\nabla_n\left(\sum_i f_i \bar{e}_i\right) = \sum_i (\partial_n f_i)\bar{e}_i + f_i \underbrace{\langle w_{ij}, n\rangle}_{C^\infty(M;\Lambda^1 M)} \bar{e}_j$$

<u>Def^{n}</u>: Fix frame $\{\bar{e}_i\}$,

$$\Omega_{ij} := -\left(d_M w_{ij} + \sum_k w_{ik} \wedge w_{kj}\right). \in C^\infty(M;\Lambda^2 M).$$

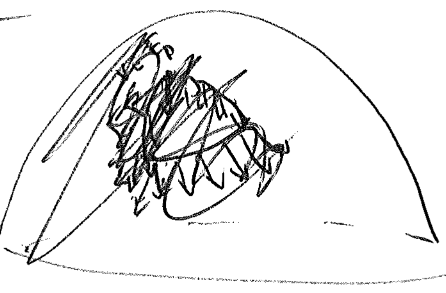
are the curvature directions for ∇ in $\{\bar{e}_i\}$.

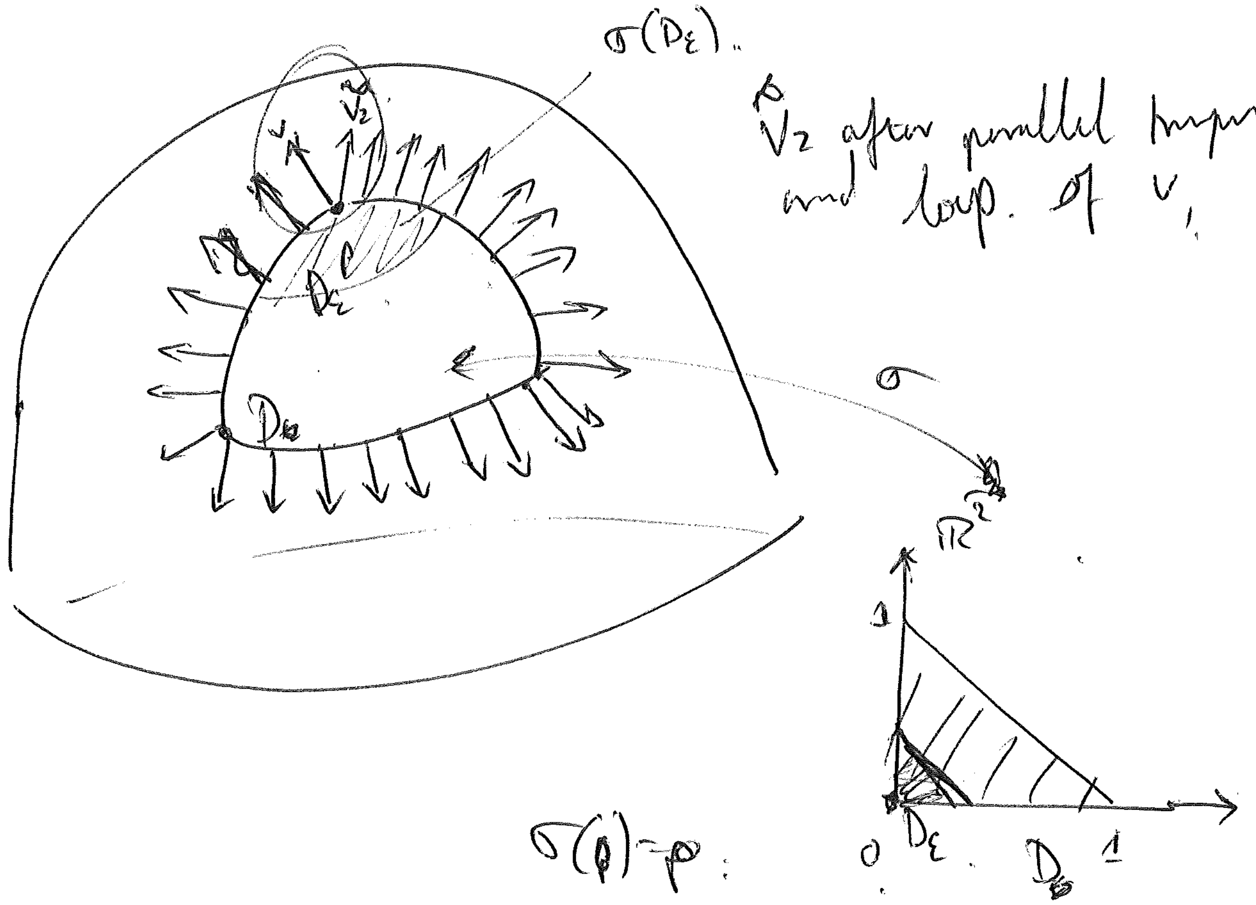
Curvature operator Ω: Given $b \in C^\infty(M;\Lambda^2 M)$

$$\langle \Omega, b\rangle \left(\sum_i f_i \bar{e}_i\right) = \sum_{ij} f_i \langle \Omega_{ij}, b\rangle \bar{e}_j .$$

inside frame $\{e_i\}$.

~~Geometric meaning of Ω:~~





Consider $v_1 \in E_p$. Do parallel transport of v_1 around $\partial\sigma(D_\varepsilon)$, obtain $v_2 \in E_p$.

Then,

$$v_2 = v_1 + \langle \Omega, \sigma_p(\underbrace{\int_{D_\varepsilon} d\hat{u}}_{\frac{\varepsilon}{2} e_1\wedge e_2}) \rangle v_1 + O(\varepsilon^3).$$

- Ω <u>does not</u> depend on the choice of frame, unlike ω_{ij}!
- $\Omega \in C^\infty(M; \Lambda^2 M \otimes \mathcal{L}(E))$.
- $\nabla_u(\nabla_v F) - \nabla_v(\nabla_u F) - \nabla_{[u,v]}F = \langle \Omega, u\wedge v\rangle F$.

<u>Defⁿ</u>: The Reimann curvature operator.

$R = \Omega$ is the curvature operator for the Levi-Civita covariant derivative on TM.

$R \in C^\infty(M; \Lambda^2 M \otimes \mathcal{L}(TM))$. $\Lambda^2 M \simeq$ skew adjoint operators,

$\downarrow$

But actually $R \in C^\infty(M; \Lambda^2 M \otimes \Lambda^2 M)$ because $R(u \wedge v) \in \mathcal{L}(TM)$ is skew adjoint and $\Lambda^2 M \simeq$ skew adjoint operators.

- Reiman curvature bivectors for a frame. $R_{ij} \in C^\infty(M; \Lambda^2 M)$.

- Riem curvature coefficients.

$$\underbrace{R_{ijkl} = \langle R_{ij}, \bar{e}_k \wedge \bar{e}_l \rangle}_{\text{Treat this as fundamental.}}$$

Prop $E = \not{\Lambda} M = \Lambda M = \Lambda(TM)$.

∇ be the L.C. cov. derivative on ΛM.

R for TM, and Ω for ΛM.

Define $R^2(b) := \sum_{i<j} \langle R_{ij}, b\rangle \, \bar{e}_i \wedge \bar{e}_j \in C^\infty(M, \Lambda^2 M)$.

R^2 ← not square, but two

$b \in C^\infty(M; \Lambda^2 M)$

Then,

$$\langle \Omega, b\rangle F = \frac{1}{2}[R^2(b), F]_\Delta = \frac{1}{2}\left(R^2(b) \Delta F - F \Delta R^2(b)\right)$$