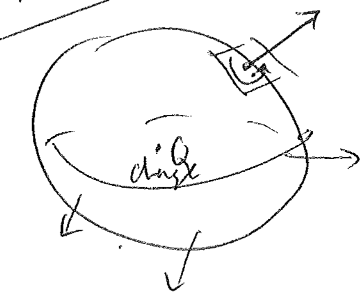
Lecture 13 20/10/2014.

$\Lambda^0\mathbb{R}^3$ **Gauss law for E.**

$$\int_{\partial D}\langle E, *d\hat{n}\rangle = \frac{1}{\varepsilon_0}Q = \boxed{\int_D \rho\, dv, \text{ charge density}}$$

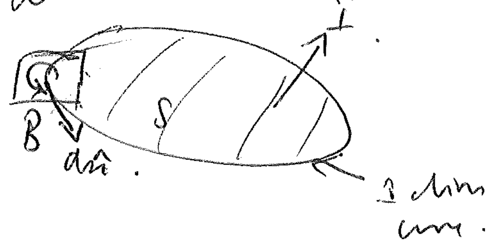
proportional constant.



$\Lambda^1\mathbb{R}^3$ **Ampère's law.**

linear dens.

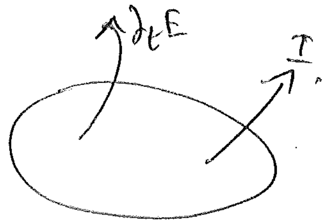
$$\frac{1}{\mu_0}\int_{\partial S}\langle B, *d\hat{n}\rangle = I = \int_S\langle J, *d\hat{n}\rangle$$



permeability $\simeq 4\pi 10^{-8}$.

Ampère – Maxwell (non static).

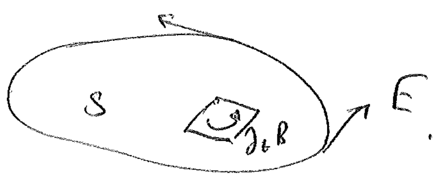
$$(+)\ \varepsilon_0\int\langle\partial_t E, *d\hat{n}\rangle$$



$\Lambda^2\mathbb{R}^3$ **Faraday's law:**

$$\int_{\partial S}\langle E, \cancel{*}d\hat{n}\rangle = -\int_S\langle\partial_t B, d\hat{n}\rangle.$$

*No *'s.



$\Lambda^3\mathbb{R}^3$ Gauss law for B

$$\int_{\partial D}\langle B, d\hat{n}\rangle = 0.$$

$$\int_{\partial D} \langle E, *d\hat{n} \rangle = \int_D \langle \nabla \wedge (*E), d\hat{n} \rangle = \int \langle (*\rho), d\hat{n} \rangle.$$

$$\int_{\partial S} \langle B, *d\hat{n} \rangle = \int_S \langle \nabla \wedge (*B), d\hat{n} \rangle.$$

$$\int_{\partial S} \langle E, d\hat{n} \rangle = \int_S \langle \nabla \wedge E, d\hat{n} \rangle$$

vector $\neq$ bivector

$$\int_{\partial D} \langle B, d\hat{n} \rangle = \int_D \langle \nabla \wedge B, d\hat{n} \rangle$$

bivector 3vector.

Four Maxwell equations (since the integral eq^n's hold $\forall$. E, B).

$\Lambda^0 \mathbb{R}^3$ $\quad \varepsilon_0 \nabla \lrcorner E = \rho$ $\quad$ (since $\nabla \wedge (*E) = *(\nabla \lrcorner E)$.)

$\Lambda^1 \mathbb{R}^3$ $\quad \varepsilon_0 \partial_t E + \frac{1}{\mu_0} \nabla \lrcorner B = -J.$

$\Lambda^2 \mathbb{R}^3$ $\quad \nabla \wedge E = -\partial_t B.$

$\Lambda^3 \mathbb{R}^3$ $\quad \nabla \wedge B = 0.$

Start with E in $\Lambda^1 \mathbb{R}^3$, B in $\Lambda^2 \mathbb{R}^3$.

Def^n. Total EM field $F = \frac{1}{c}E + B$. $~~F = \frac{1}{c}(E+B)~~$ $\in \Lambda^1\mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3 \subset \Lambda \mathbb{R}^3$.

where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$, speed of light.

Rescale eq^{n}s.

$\Lambda^0\mathbb{R}^3$ $\quad \varepsilon_0 \nabla \lrcorner \left(\frac{1}{c}E\right) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\rho$.

$\Lambda^1\mathbb{R}^3$ $\quad \frac{1}{\mu_0}\left(\frac{1}{c}\partial_t\right)\left(\frac{1}{c}E\right) + \frac{1}{\mu_0}\nabla \lrcorner B = -J$.

$$\frac{1}{c}\partial_t\left(\frac{1}{c}E\right) + \nabla \lrcorner B = -\mu_0 J.$$

$\Lambda^2\mathbb{R}^3$ $\quad \frac{1}{c}\partial_t B + \nabla \wedge \left(\frac{1}{c}E\right) = 0$.

$\Lambda^3\mathbb{R}^3$ $\quad \nabla \wedge B = 0$.

$+$

$$\frac{1}{c}\partial_t F + \mathbb{D}F = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\rho - \mu_0 J = j.$$

$$\mathbb{D}F = \nabla \wedge F + \nabla \lrcorner F.$$

So, Maxwell $\quad \frac{1}{c}\partial_t F + \mathbb{D}F = j$

$$\left(\frac{1}{c}\partial_t - \mathbb{D}\right)\left(\frac{1}{c}\partial_t + \mathbb{D}\right)F = \frac{1}{c}(\partial_t - \mathbb{D})j$$

$$= \frac{1}{c}\partial_t^2 - \mathbb{D}^2 = \frac{1}{c}\partial_t^2 - \Delta$$

lose information has gone to 2nd order.

$\Rightarrow$ Maxwell is a wave eq^n for each 6 components.
+ 1st order coupling

3

Dirac evolution eq^{n}s in 4-dim Spacetime W.

Fix basis e_0, e_1, e_2, e_3.

$\langle e_0, e_0\rangle = -1, \quad \langle e_i, e_i\rangle = +1.$

Two natural Dirac operators: $\mathbb{D}_W$ (Hodge), $\not{\mathbb{D}}_W$ (Spin)

$\mathbb{D}_W F = \nabla \wedge F$ on $F : W \to \Delta W$ $\quad$ dim $\Delta W = 16$.

$\not{\mathbb{D}}_W \psi = \nabla \cdot \psi \qquad \psi : W \to \not{\Delta} W$

$$\mathbb{D}_W F = (\underbrace{-e_0}_{} \partial_0 + e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3) \wedge F.$$

really e_0^*, but $\langle e_0, e_0\rangle = -1 \Rightarrow e_0^* \sim -e_0$.

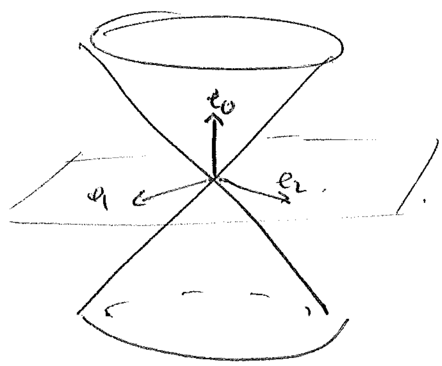
~~For Spin Dirac,~~ Recall $\exists$ alg. ~~homom~~ isomorphism.

$\Delta \mathbb{R}^3 \approx \Delta^{ev} W$.

$e_i \mapsto e_0 e_i$

$e_i^2 = +1$.

$(e_0 e_i)(e_0 e_i)$
$= -e_0 e_0 e_i e_i$
$= (+1)(+1)$
$= +1$.



Translate ~~maxw~~ maxwell

$$\frac{1}{c}\partial_t + e_0 \mathbb{D} = \frac{1}{c}\partial_t + e_0\left(\sum e_i \partial_i\right)$$

$$F = \frac{1}{c}E + B \longmapsto \frac{1}{c} e_0 E + B.$$

$$j \longmapsto \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\rho - \mu_0 e_0 J.$$

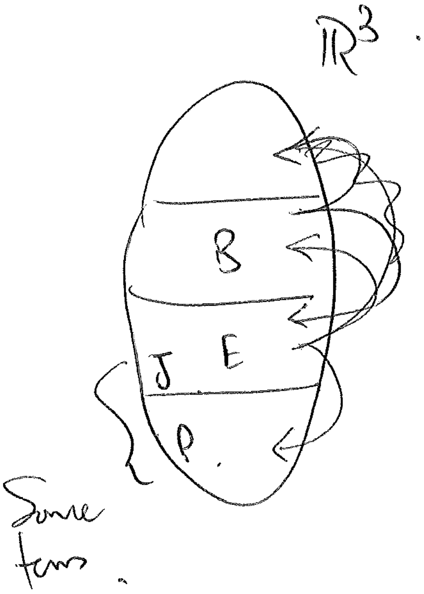
Ie, Maxwell

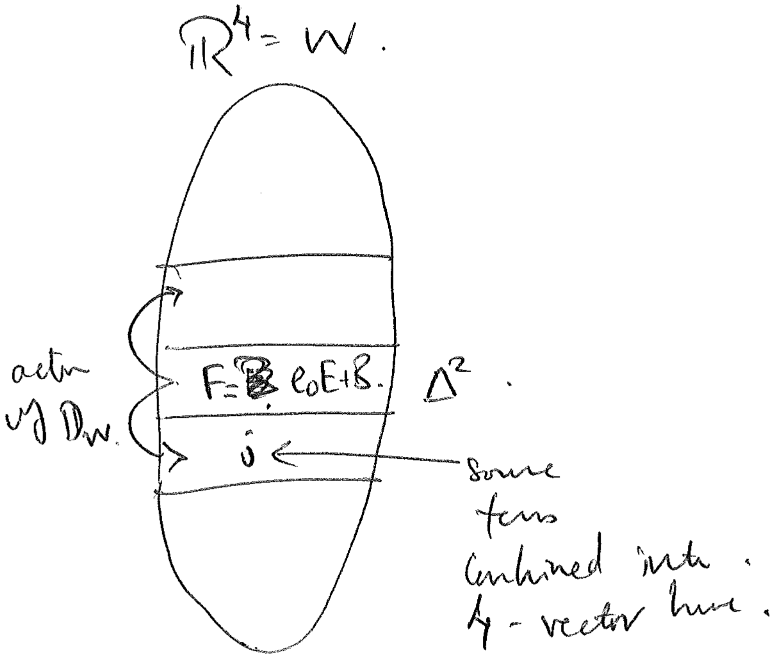
$$\frac{1}{c}\partial_t + e_0\left(\textstyle\sum_i e_i \partial_i\right)\left(\frac{1}{c} e_0 E + B\right) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\rho - \mu_0 e_0 J.$$

Multiply by $-e_0$:

$$D_W\Big(\underbrace{\tfrac{1}{c}e_0E+B}_{\text{bivector, } \Delta^2 W \text{ - 6 dim.}}\Big) = \overbrace{-\sqrt{\frac{\mu_0}{\varepsilon_0}}\,e_0\rho - J}^{\text{v. field in } W.}.$$

Conclusion to discussion.





So, write Maxwell's eqns as
$$\begin{cases} d_W F = 0 \\ \delta_W F = j \end{cases}$$

We will see that $d_W F = 0 \iff F = d_W A$; A v. field.

This A is non-unique! Highly non unique!

Even though $F = d_W A$ is non-unique, it is more fundamental $\longrightarrow$ Experiment of Bohn – Aheronov.

Dirac's eq^n for the electron.

Potential from before, $F = d_W A$.

$$\hbar \not{D}_W \psi = mc^2\psi + i q A \cdot \psi .$$

Planck's constant. electron mass. electron charge.

$|\psi|^2$ is indep of A — this is the important feature of ψ, it is the prob density.

$\psi : W \to \not{S} W$ Spinor field "wave function" for the electron!

$$\not{D}_W = -e_0 \cdot \partial_0 + e_1 \cdot \partial_1 + e_2 \cdot \partial_2 + e_3 \cdot \partial_3 . \quad \text{Spin-Dirac.}$$

<u>Example</u>. Free electron:

$$(*) \quad \hbar \not{D}_W \psi = mc^2 \psi .$$

$$\Rightarrow \quad \hbar^2 \not{D}_W^2 \psi = \hbar \not{D}_W mc^2 \psi = m^2 c^4 \psi .$$

$$\hbar^2 \not{D}_W^2 \psi = \hbar^2(-\partial_0^2 + \Delta)\psi . \qquad \text{Klein-Gordon eq}^n.$$

Dirac started with this, and wanted to solve for $(*)$. This is what he did.
Andreas said "all on his tombstone."

On the physical meaning of ψ.

Classically, $|\psi|^2$ = ~~pd~~ probability density.

But we need $\not\Lambda W$ to have a norm, so, what is this?

ψ gives a "probability current" $j_p : W \to \Lambda^1 W$.

s.t. $\langle j_p, v \rangle = D(\psi, v \cdot \psi)$

where v is another v-field, and D the the Dirac form:

(I) D sesquilinear.

(II). D satisfies $D(\psi_1, \hat{w} \cdot \psi_2) = D(\overline{w}^c \cdot \psi_1, \psi_2)$.

Ie, compatible with action on Cliff alg,

Note: $\overline{w}^c$, expect reversion. But we also need involution.

? Uniqueness & Existence of such a D?

Fix std rep of $\not\Lambda W$ and write

$$D(\psi_1, \psi_2) = \psi_1^* M \psi_2 .$$

$$\Rightarrow \quad M \rho(\hat{w}) = (\rho(\overline{w}^c))^* M .$$

$$\Leftrightarrow \quad M \rho(w) = (\rho(\hat{\overline{w}}^c))^* M .$$

$$\Leftrightarrow \quad \underbrace{\rho(w)}_{\text{representation of } \Lambda W \text{ on } \not\Lambda W} = M^{-1} (\rho(\hat{\overline{w}}^c)^*) M .$$

as is

From before, such rep. are unique upto $\mathbb{C}$-scalars.

$\Rightarrow$ M exists and is unique upto $\mathbb{C}$-scalars.

Calculation of M: Remove lecture (6?).

$$\Rightarrow \quad \rho(e_0) = \begin{bmatrix} & & -i & \\ & & & -i \\ -i & & & \\ & -i & & \end{bmatrix}, \quad \rho(e_2) = \begin{bmatrix} & & 1 & \\ & & & -1 \\ & 1 & & \\ -1 & & & \end{bmatrix}.$$

$$\rho(e_1) = \begin{bmatrix} & & -i & \\ & & & -i \\ & i & & \\ i & & & \end{bmatrix}, \quad \rho(e_3) = \begin{bmatrix} & & -i & \\ & & & i \\ i & & & \\ & -i & & \end{bmatrix}$$

Need to figure out $e_i \longmapsto \hat{e}_i^c$.

$\hat{e}_i = -e_i$.

$\rho(\hat{e}_0^c)^* = -\rho(e_0)^* = +\rho(e_0)$.

$\rho(\hat{e}_i^c) = -\rho(e_i)^* = \cancel{\rho(e_i)} - \rho(e_i)$. $\quad i = 1, 2, 3.$

$\Rightarrow$ $\underline{M = \rho(e_0)}$ works.

Define. $D(z_1, z_2) = z_1^* \begin{bmatrix} & & -i & \\ & & & -i \\ -i & & & \\ & -i & & \end{bmatrix} z_2$.

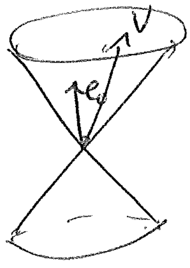
Because we've fixed rep, we have this. But indep of rep means that this is indep of basis and rep.

- D is skew-symm,

$$D(\psi_1, \psi_2) = -D(\psi_2, \psi_1)^c .$$

- For $D(\psi, v\cdot\psi) > 0$.

real, future pointing time like vector.



- Check $D(\psi, e_0 \cdot \psi) = -\psi^* \underbrace{\rho(e_0)\rho(e_0)}_{\rho(e_0^2)} \psi = |\psi|^2 > 0$.

Back to probability current

$$\langle j_p, v\rangle = D(\psi, v\cdot\psi) .$$

$$F \xrightarrow{\text{Solve for}} A \xrightarrow{\text{Solve for Dirac}} \psi \longrightarrow \underline{j_p} .$$

Want: $\langle j_p, e_0\rangle = \rho_p$, probability density ie how likely it is that the electron is in a certain region.

$$\langle j_p, e_0\rangle = D(\psi, e_0\cdot\psi) = |\psi|^2 > 0 .$$