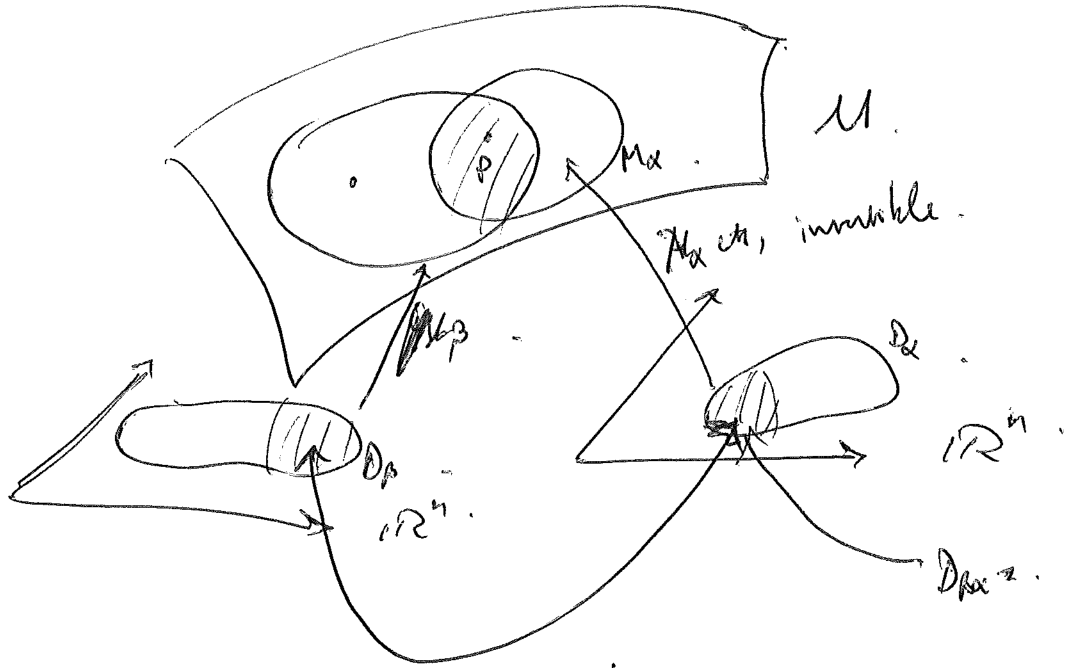
~~16/10/2014.~~ Lecture 12. 16/10/2014.

Manifolds. n-dimensional, smooth.

- Abstractly: A top. space with $\mathbb{R}^n$ geometry locally.



$\mu_{\beta\alpha} = \mu_\beta^{-1} \circ \mu_\alpha$.

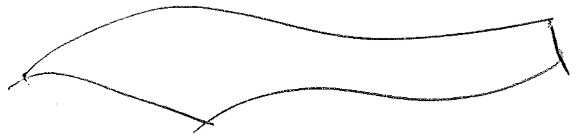
Concretely: A family of transition maps ~~[illegible]~~.

$\mu_{\beta\alpha} : D_{\beta\alpha} \to D_{\alpha\beta}$.

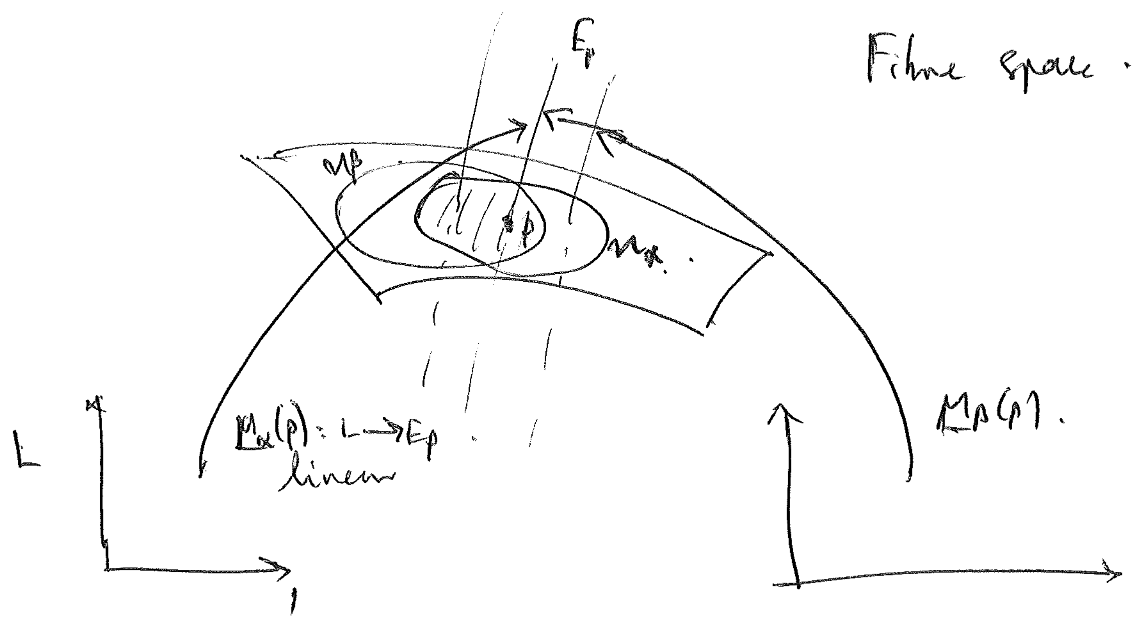
which are <u>smooth</u>.

Vector bundles. (bundles of linear spaces, or assoc. algebras).

- Abstractly: Family of linear spaces $E = (E_p)_p$ depending smoothly on p.



Fibre space . $L = \mathbb{R}^n$ or $\mathbb{C}^n$.



$\underline{\mu}_{\beta\alpha}(p) : L \rightarrow L$. linear.

$$M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha .$$

Conversely: a Family of bundle transition maps.

$$\underline{\mu}_{\beta\alpha} : \mathcal{U}_\beta \cap \mathcal{U}_\alpha \longrightarrow \mathcal{L}(L).$$

Note: $\underline{\mu}_{\beta\alpha} = \underline{\mu}_{\alpha\beta}^{-1}$

Example: the tangent bundle. TM

$$\mu_{\beta\alpha} : D_{\beta\alpha} \longrightarrow D_{\alpha\beta} .$$

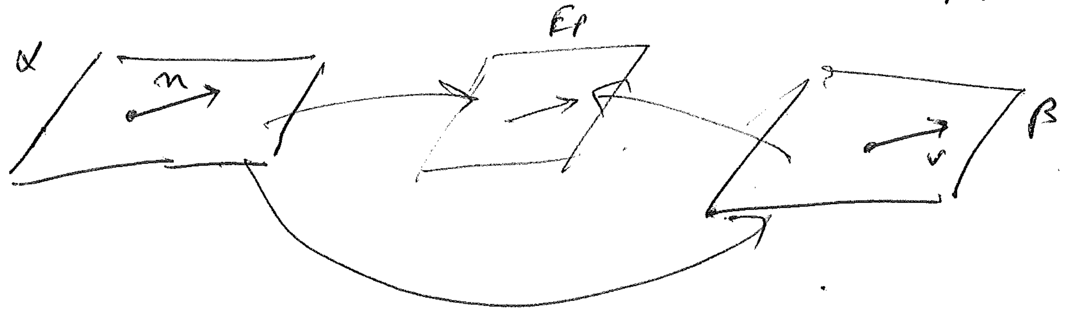
Jacobian $\underline{\mu_{\beta\alpha}}_{x} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.
$\quad \hat{D}_{\beta\alpha}$

Then TM is the vector bundle with transition maps.

$$\underline{\mu}_{\beta\alpha}(p) := \underline{\mu_{\beta\alpha}(x)} \qquad p = \mu_\alpha(x)$$

Fibre space $L = \mathbb{R}^n$

Fibre at $p \in M$; $E_p = T_pM := \{ (\alpha, u) \in I \times \mathbb{R}^n$ with $(\alpha, u) \sim (\beta, v)$ if $v = \mu_{\beta\alpha}(u). \}$.



Example: The exterior bundle ΛM over a given manifold.

Fibre space $L = \Lambda \mathbb{R}^n$

Fibre at $p \in M$: $E_p = \Lambda(T_pM)$.

Transition maps: $\mu_{\beta\alpha} : \Lambda\mathbb{R}^n \longrightarrow \Lambda\mathbb{R}^n$.

"matrix of subdeterminants of the matrix of 1st order derivatives of $\mu_{\beta\alpha}$."

- A section of a vector bundle E is "an E-valued function on M."

$$w : M \to E$$
$$w(p) \in E_p.$$

- We write $C^\infty(M; E)$ smooth sections, $L^2(M; E)$ L^2-section etc.

- A frame for E in $U \subset M$ is a set of N sections of E in U.

$$\bar{e}_1(p), \dots, \bar{e}_N(p),$$

$p \in U$ s.t. $E_p = \text{span}\{\bar{e}_1(p), \dots, \bar{e}_N(p)\}$.

(Cannot have global smooth frame, ie $U = M$ in general.)

- Example. Coordinate frame for TM.

$$\bar{e}_i(p) = M_x(e_i)$$ ← stand basis in $\mathbb{R}^n$.

In general, non - ON!

A metric on a bundle E is an inner product

$\langle u, v \rangle_p$ on each fibre E_p,

depending smoothly on p.

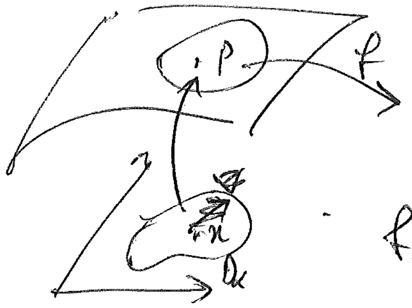
We always assume Euclidean inner products in the manifold setting

$\langle \cdot, \cdot \rangle$ Reim. is when $\langle \cdot, \cdot \rangle > 0$ and symm.

Derivatives

(1) Scalar function; $f : M \to \mathbb{R}$

$f(p)$ on M.



$f_\alpha(x) = f(\mu_\alpha(x))$.

vector v at x, partial derivative.

$$\partial_v f_\alpha(x) =: \partial_{\mu_\alpha(v)} f(p)$$ ← indep of α by chain rule.

where $\mu_\alpha(v) \in T_pM$.

(2) vector field; $v \in C^\infty(M; TM)$. = section of TM.

Prop. (M, g) Riem. mfld. $\exists!$ covariant derivative on TM.

$$(\nabla_n v)(p) \in T_pM$$

for $n \in T_pM$ such that

(linear in n, addit. and satisf. Leibniz rule in v)

(I). $$\partial_n \langle v_1(p), v_2(p)\rangle_p = \langle \nabla_n v_1(p), v_2(p)\rangle_p + \langle v_1(p), \nabla_n v_2(p)\rangle_p.$$

(II). $$\nabla_{n(p)} v(p) - \nabla_{v(p)} n(p) = [n(p), v(p)].$$

Calculation in a frame: $\bar{e}_i(p)$ for TM

$$v(p) = \sum_i f_i(p)\,\bar{e}_i(p).$$

$$\begin{aligned}\nabla_u(v(p)) &= \sum_i \nabla_u(f_i(p)\,\bar{e}_i(p)) \\ &= \sum_i \partial_u f_i(p)\,e_i(p) + f_i(p)\nabla_u\bar{e}_i(p)\end{aligned}$$

Notation due to Cartan:

$$\nabla_u \bar{e}_i(p) = \sum_j \underbrace{\langle w_{ji}(p), u\rangle}\,\bar{e}_j(p).$$

regard w_{ji} as vector fields rather than co-vectors. ~~$\sum$ $w_{ji}(p)(u)$ depends linearly on u, but just # isom.~~

For fixed $\{\bar{e}_i(p)\}$, ∇ is uniquely by the Christoffel symbols / connection 1-forms.

Typically, ~~$\nabla_u \bar{e}_i(p)$~~ Cartan writes:

$$\nabla_u \bar{e}_i(p) = \sum_j w_{ji}(p)(u)\,\bar{e}_j(p).$$

where $w_{ji} \in C^\infty(M; T^*M)$, ie 1-form.

But $w_{ji} \longmapsto \langle w_{ji}, \cdot\rangle = w_{ij}^\flat$ to obtain 1-vector.

Exercise 6.33.

- $\{\bar{e}_i\}$ coord. $\Rightarrow$ w_{ij} determined by 1st order derivatives of metric.

- $\{\bar{e}_i\}$ ON-frame $\Rightarrow$ w_{ij} determined by ~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~ $[\bar{e}_i, \bar{e}_j]$.

③ Derivations of multi-vector fields.

$$w \in C^\infty(M; \Lambda M).$$

$\exists!$ Covariant derivative (Levi-Civita) on ΛM.

s.t.

(I). $\nabla_n f = \partial_n f$, $f \in C^\infty(M; \Lambda^0 M)$

(II) $\nabla_n v =$ Levi-Civita $v \in C^\infty(M; \Lambda^1 M)$.

(III) $\nabla_n (w_1 \wedge w_2) = (\nabla_n w_1) \wedge w_2 + w_1 \wedge \nabla_n w_2$.

Lie Derivations

- M Riem. $\Rightarrow$ $\nabla_n v(p)$ only uses value of n at p.
- M gen. smooth.; there is a notion of derivative along a vector field $n(p)$, Lie Derivative. $\mathcal{L}_{n(p)} v(p)$.

the idea:

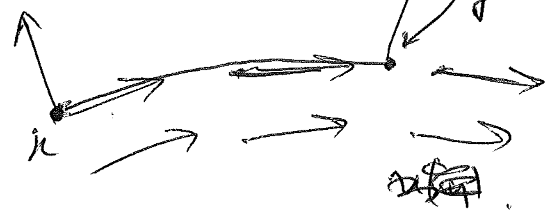
- constant vector: $u \in V$, V vector space.

  $\Rightarrow$ translation. $F_u^t : x \mapsto x + tu$.

- v. field, $u(x)$ in coord chart, gives a flow. $F_u^t : x \mapsto F^t(x)$.

  by soln ODE: good properties by std theory.

  So, F^t: local diff.

- 

  $V(y)$, $y = F_u^t(x)$ via ~~flo~~ flow.

  v. field u.

Fix t so ~~Take~~ that ~~inverse~~ F^{-t} fn. $F_u^t(y) = y$.

to move. $\underline{F_u^{-t}}_y (V(y))$.

Here, $\underline{F_u^{-t}}$ is the Jacobian,

$\underline{F_u^{-t}}_y : T_yM \to T_xM$. So,

$\underline{F_u^t}_y (V(y)) \in T_xM$. or $\underline{F_u^t}_x (V(x)) \in T_yM$.

- $\mathcal{L}_{u(x)} V(x) = \frac{d}{dt}\Big|_{t=0} \underline{F_u}^{t}_{x} (V(x)) = [u, V]$.