13/10/2014.

(X, V) inner product space

$\Lambda V = \Delta V$ cliff. algebra $\ni w$

$\not\Delta V$ = complex spinor space. $\ni \psi$.

$w : \psi \longmapsto w \cdot \psi$.

Hodge / Clifford Dirac Op.

$$\nabla \vartriangle F = \sum_i e_i^* \vartriangle \partial_{x_i} F(x) =: \mathbb{D} F.$$

Atiyah-Singer / Spin Dirac Op.

$$\nabla \cdot \psi = e_i^* \cdot \partial_{x_i} \psi(x) =: \not{\mathbb{D}} \psi.$$

action of vectors on spinors via representation.

(A) Euclidean Dirac Operators.

$\{e_i\}$ ON. basis : $e_i^* = e_i$.

$$\mathbb{D}^2 F = \sum_{ij} e_j e_i \partial_{x_j} \partial_{x_i} F$$

$$= \sum_{ij} \tfrac{1}{2}(e_i e_j + e_j e_i) \partial_{x_j} \partial_{x_i} F = \sum_i \partial_{x_i}^2 F = \Delta F$$

Componentwise

(Remember, ordinarily, this would be Hodge-Laplace, but we're in Euclidean space, so this is exactly ~~Hodge~~ Laplace acting componentwise.

Similarly, $\not{\mathbb{D}}^2 = \Delta$.

$\mathbb{D}$, $\not{\mathbb{D}}$ are "Dirac type ops."

I.e., 1st order PDO $= \sqrt{\Delta}$.

But this is not a diff op. In order to obtain diff op, need to enlarge space to obtain a differential square root.

Example. $\mathbb{D}$ generalises Cauchy-Riemann.

$$V = \mathbb{R}^2 = \mathbb{C} \qquad f : \Omega \subset \mathbb{C} \to \mathbb{C} = \Delta^{ev} V = \Delta^0 V \oplus \Delta^2 V.$$

$$f(z) = u(z) + j v(z), \qquad j = e_{12}.$$

$$\mathbb{D} f = (e_1 \partial_x + e_2 \partial_y) \vartriangle (u + j v)$$
$$= (\partial_x u - \partial_y v) e_1 + (\partial_y u + \partial_x v) e_2.$$

$$\mathbb{D} f = 0 \iff f \text{ hol.}$$

Ex. $\not{\mathbb{D}}$: $V = \mathbb{R}^2 = \mathbb{C}$. $\not{\Delta} V = \mathbb{C}^2 \ni \psi$.

$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$. Std. rep. $e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

$$0 = \not{\mathbb{D}} \psi = (e_1 \partial_x + e_2 \partial_y) \psi$$
$$= \begin{bmatrix} 0 & \partial_x - i \partial_y \\ \partial_x + i \partial_y & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \Rightarrow \begin{array}{l} \psi_1 \text{ analytic} \\ \psi_2 \text{ anti-analytic} \end{array}$$

Example: V = n-dim Euc. space.

$\mathbb{D}F = 0$ for $F : V \to \Lambda^1 V$ (i.e. v.field).

$$\mathbb{D}F = \nabla \Delta F = \underset{\substack{\uparrow \\ \text{inner prod.}}}{\nabla \cdot F} + \nabla \wedge F$$

$$= \operatorname{div} F + \operatorname{curl} F \in \Lambda^0 V \oplus \Lambda^2 V.$$

So, $0 = \mathbb{D}F \iff \begin{cases} \operatorname{div} F = 0 \\ \operatorname{curl} F = 0. \end{cases}$

(Stein-Weiss Fields - arises in conjunction $\mathbb{R}$-Hardy spaces).

close connection to harmonic functions since:
curl $F = 0 \Rightarrow F = \nabla u$ and div $F = 0 \Rightarrow \Delta u = 0$.

$\underline{\Delta u = \operatorname{div}(\nabla u) = 0}$ ⟸ since.

Def^{n}. $F : \Omega \subset V \to \Delta V$.

If $\mathbb{D}F = 0$, then say F is monogenic, and similarly $\not{\mathbb{D}} \psi = 0$ is monogenic for $\psi : \Omega \to \Delta V$.

(left monogenic here).

Comparison to classical function theory.

① If $f(z)$ & $g(z)$ are analytic, then so is $f(z)\,g(z)$.

$$\nabla \triangle \left(F(x) \triangle G(x) \right).$$

$$= \nabla \triangle \left(\dot{F}(x) \triangle G(x) \right) + \underbrace{\nabla \triangle \left(F \triangle \dot{G} \right)}_{\sum_i e_i \triangle F(x) \triangle \partial_{x_i} G(x)}$$

$$= \underbrace{(\nabla \triangle F)}_{0 \text{ if } f \text{ analytic}} \triangle G$$

~~non~~- commutative.

So, not true for monogenic fields.

However, if $\mathbb{D}F = 0$, then $\mathbb{D}(F(x) \triangle w) = 0$.

② If $f(z)$ and $g(z)$ are analytic, then so is $f(g(z))$.

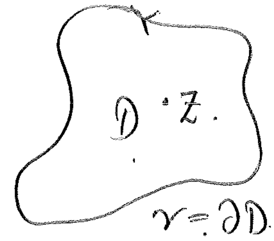
If $\mathbb{D}F = 0$, then it can be pulled back through a conformal map $g: V \to V$ via a Kelvin-Transform.

"$F(g(x))$"

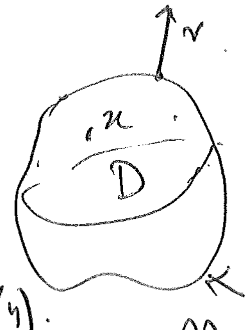
but factor in front.

(3) If $f(z)$ is analytic, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw.$$



Thm. If $F : D \to \triangle V$ is monogenic is monogenic. then



$$F(x) = \int_{\partial D} \frac{y-x}{\sigma_{n-1}|y-x|^n} \triangle \nu(y) \triangle F(y)\, d\sigma(y).$$

∂D smooth.

Pf. Recall Stokes' thm for $f(y,w)$ when w is a 1-form (linear). Write $E(y-x) = \dfrac{y-x}{\sigma_{n-1}|y-x|^n}$.

Define $f(y,w) = E(y-x) \triangle \nu \triangle F(y)$.

Stokes $\displaystyle\int_{\partial D} f(y, \nu(y))\, d\sigma(y)$

$$= \int_D f(\dot{y}, \nabla)\, dy.$$

$$\underset{\textstyle \Sigma_i e_i \partial_i}{f(\dot{y}, \nabla)} = \sum_i \partial_{y_i} E(y-x) \triangle e_i \triangle F(y) + \sum_i E(y-x) \triangle \cancel{\left(e_i \triangle \partial_{y_i} F(y)\right)}_{=0}$$

by monogenic ass.

$$= \operatorname{div}_y E(y-x) F(y) - (\operatorname{curl}_y E(y-x)) \triangle F(y).$$

(since $E \triangle \nabla = \langle E, \nabla \rangle + (-\nabla \wedge E)$).

$$= \delta_x(y) F(y) \quad \text{since } D(\underbrace{D\Phi}_{\nabla\Phi = E}) = \delta_0.$$

Φ fund. sol^n to Δ.

(4) Corollary: monogenic fields are $\mathbb{R}$-analytic.

For analytic functions, $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

Similarly, monogenic fields can be expanded in Taylor series

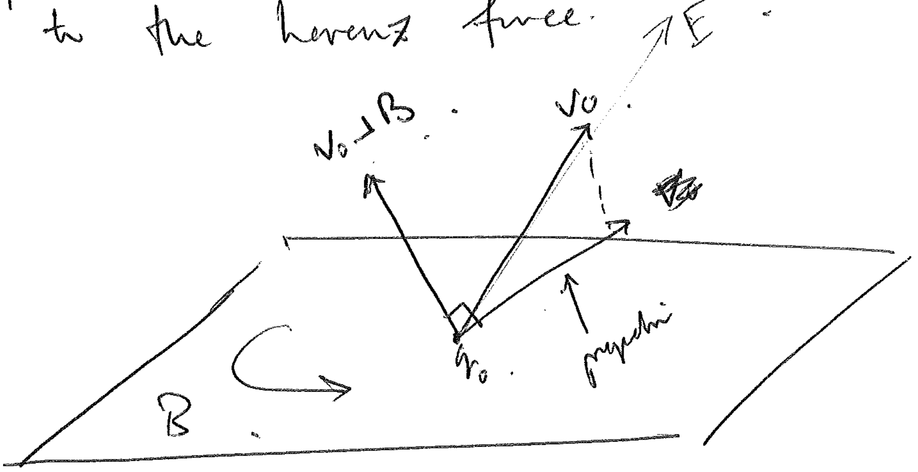
Maxwell's Equations

In 3D space, we have: (universe).

(1) a electric field, $E(t,x) \in \Lambda^1 \mathbb{R}^3$. v. field.

(2) magnetic field, $B(t,x) \in \Lambda^2 \mathbb{R}^3$ bi-vector. (unlike what usually people say)

They influence a charge q_0 with velocity v_0 according to the Lorenz force.

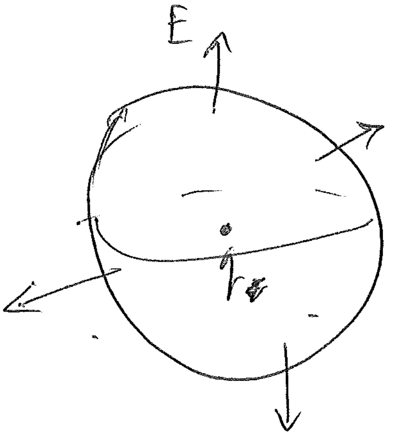


$$F = q_0 (E - v_0 \lrcorner B) = q_0 (E + B \llcorner v_0).$$

Classical notation $F = q_0 (E + v_0 * (*B))$.

Maxwell's equations are a sort of inverse. I.e., Conversely, give time evolution of E & B given charges in space.

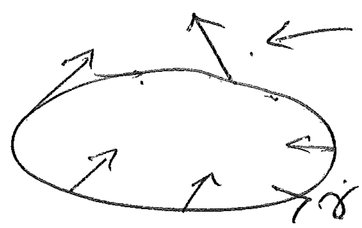
(1) Electrostatics



Gauss law.

$\boxed{\Lambda^0 \text{ eq}^n}$

E is conservative.

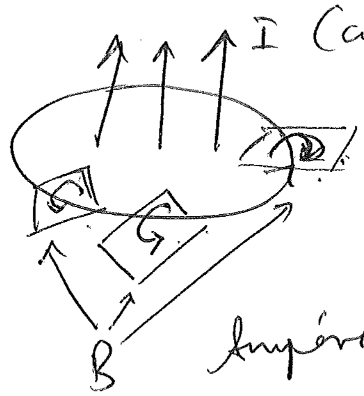


cannot be tangential everywhere.

$\oint_\gamma E \cdot d\hat{n} = 0.$

$\boxed{\Lambda^2 \text{ eq}^n}$

(2) Magnetostatics

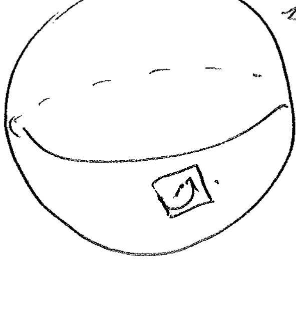


$\boxed{\Lambda^1 \mathbb{R}^3 \text{ eqn.}}$

Ampère's law.

Steady electric current gives a "swirling" B-field. B is a closed bivector field.

$\int_{\partial D} \langle B, d\hat{n} \rangle = 0.$



$\boxed{\Lambda^3 \mathbb{R}^3 \text{ eq}^n}$