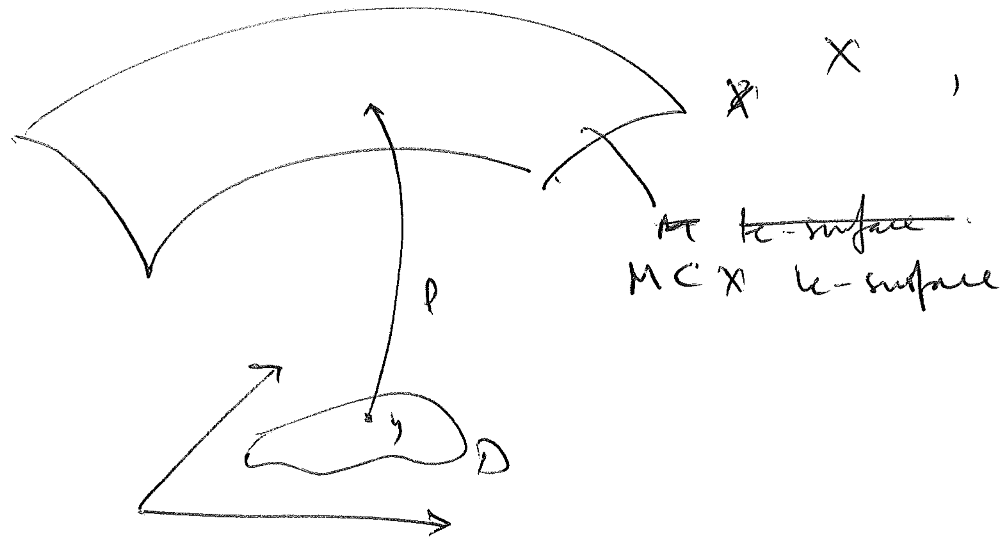
Lecture 10 09/10/2014.

Generalised notion of k-vectors.

(X, V) affine space, $\dim V = n$.



Assume given a parametrisation. $p : D \subset \mathbb{R}^k \to M \subset X$.

e_k = std orientation of $\mathbb{R}^k$

$\underline{p}_y : \mathbb{R}^k \to T_x M$.

$\underline{p}_y : \wedge \mathbb{R}^k \to \wedge (T_x M) \subset \wedge V$.

$\underline{p}_y (e_k) \leftarrow$ orientation of M at x.

Def^n General k-form in X is a function

$f(x, w) \in L$ = fixed linear space (not nec. $\mathbb{R}$V).

$x \in X$, $w \in \hat{\wedge}^k V$.

Such that $f(x, \lambda w) = \lambda f(x, w)$. $\forall \lambda > 0$.

The integral of such f over M is:

$$\int_M f(x, d\hat{n}) := \int_D f(\rho(y), \rho_{*y}(e_{\hat{\imath}}))\, dy.$$

$\hat{\Lambda}^n V$.

We note that $\int_M f(x, d\hat{n})$ is independent of ρ but in general dependent on orientation of M.

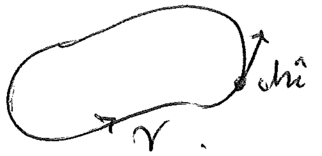
<u>Examples</u>:

(1) $M = \mathbb{R}^k$;

$$\int_{\mathbb{R}^n} g(x)\, dx = \int_M f(x, d\hat{n}) \quad \text{with}$$

$$f(x, w) := g(x).\,|w|.$$

(2) $M = \mathbb{C} = \mathbb{R}^2$.

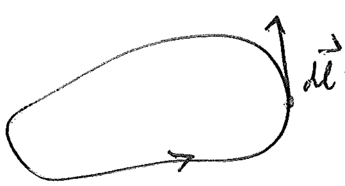


$$\int_\gamma g(z)\, dz = \int_\gamma f(x; d\hat{n}).$$

$\mathbb{C}$ multiplication.

$$g : \mathbb{C} \to \mathbb{C}, \quad f(x, w) = g(z) \cdot w$$

(3). $M = \mathbb{R}^3$ vector calculus.



dot prod.

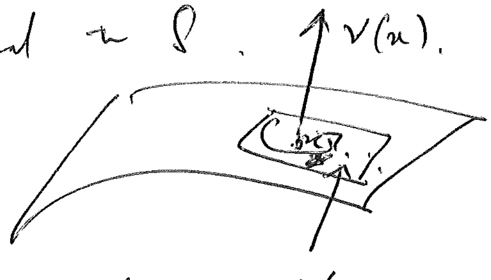
$$\int_\gamma v(x) \cdot d\vec{\ell} = \int_\gamma f(x, d\hat{n}).$$

$$f(x, w) = v(x) \cdot w$$

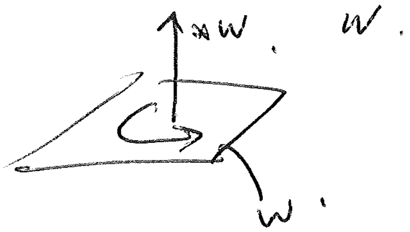
dot prod.

(4). Surface integral in vector calculus.

$\int_S v(x) \cdot \nu(x)\, dS$, $\nu(x)$ - normal to S.



$= \int_S f(x, d\hat{u})$.



$f(x, w) := \cancel{\ } v(x) \cdot (*w)$, $*w$

$\uparrow$

2-form.

<u>Note</u>: (2) - (4) are linear k-forms, but <u>not</u> (1).

<u>~~L~~inear</u> means: ~~$\wedge^k V \ni w \mapsto$~~ $\wedge^k V \ni w \longmapsto f(a, w) \in L$.

should be linear. I.e., no longer just on Grassmann ~~k-form~~ k-cone.

Fix L-basis $v_1, \ldots, v_\ell$, then for f linear:

$$f(x, w) = \sum_i \langle \theta_i(x), w \rangle v_i$$

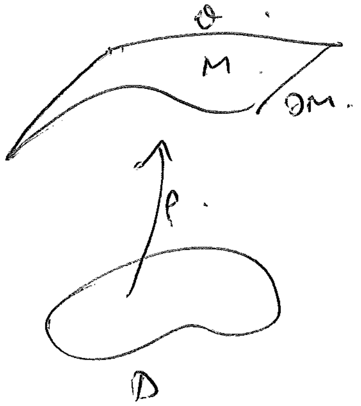
<u>Stoke's Theorem</u>

M = k-surface in X. θ $(k-1)$ covector field in a nbh of M.

$$\int_{\partial M} \langle \theta(x), d\hat{x}_{\partial M} \rangle = \int_M \langle \nabla \wedge \theta(x), d\hat{x}_M \rangle .$$

$\uparrow$ oriented meas on body. $\uparrow$ oriented measure on M.

$$\int_M \langle \nabla\wedge\theta(x), d\hat{x}_M\rangle \overset{\text{by def}^n}{=} \int_D \langle \nabla\wedge\theta(\rho(y)), \underline{\rho}_y(e_{\bar{u}})\rangle \, dy .$$

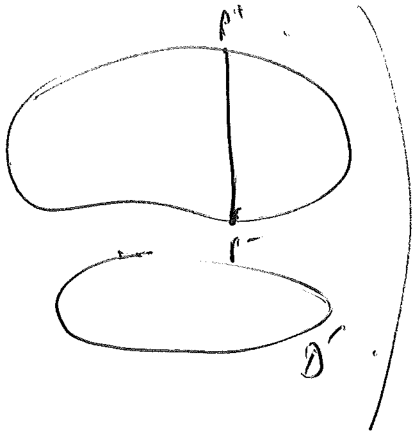


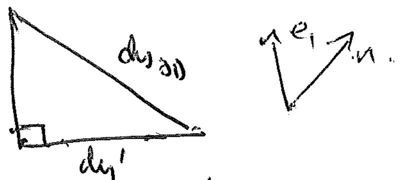
$$= \int_D \langle \rho^*(\nabla\wedge\theta)(y), e_{\bar{u}}\rangle \, dy .$$

$$= \int_D \langle \nabla\wedge(\rho^*\theta)(y), e_{\bar{u}}\rangle \, dy .$$

$$= \sum_i \int_D \partial_{y_i} \langle \rho^*\theta(y), e_i \lrcorner e_{\bar{u}}\rangle \, dy .$$

$$= \int_{D'} \left(f_1(\rho^+) - f_1(\rho^-)\right) dy' + \cdots$$





$$\frac{dy'}{dy_{\partial D}} = \langle n, e_1\rangle .$$

$$= \int_{\partial D} \langle \rho^*\theta(y), e_1 \lrcorner e_{\bar{u}}\rangle \langle n, e_1\rangle \, dy_{\partial D} + \cdots$$

(I). $\Big[= \int_{\partial D} \langle \rho^*\theta(y), n \lrcorner e_{\bar{u}}\rangle \, dy_{\partial D}$

(since $[w] = [e_{\bar{u}}] \oplus [n]$.)

$$= \int_{\partial D} \langle \rho^*\theta, n \lrcorner e_{\bar{u}} \, dy_{\partial D}\rangle .$$

$$= \int_{\partial D} \langle \rho^*\theta, d\hat{y}_{\partial D}\rangle .$$

$$= \int_{\partial D} \langle \theta(\rho(y)), \underline{\rho}_y(d\hat{y})\rangle .$$

$$= \int_{\partial M} \langle \theta(x), d\hat{x}_{\partial M}\rangle .$$

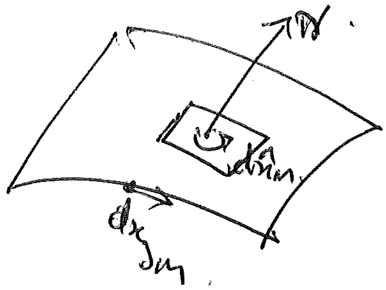
Remark: We need $d\hat{x}_{\partial M}$ and $d\hat{x}_M$ to be compatible. This was used in (I). The assumption that we implicitly make in the hypothesis is

$$\nu \wedge \frac{d\hat{x}_{\partial M}}{|d\hat{x}_{\partial M}|} = \frac{d\hat{x}_M}{|d\hat{x}_M|} \iff \frac{d\hat{x}_{\partial M}}{|d\hat{x}_{\partial M}|} = \nu \lrcorner \frac{d\hat{x}_M}{|d\hat{x}_M|}.$$

Example: Classical Stokes' thm in $\mathbb{R}^3$:

$\theta = F(x)$, v. field. Then

$$\nabla \wedge F = *(\nabla \times F)$$



$$\int_{\partial M} \langle F, d\hat{x}_{\partial M} \rangle = \int_M \langle \nabla \wedge F, d\hat{x}_M \rangle.$$

$$= \int_M \langle *(\nabla \wedge F), * d\hat{x}_M \rangle. \qquad * \text{ is an isometry}$$

$$= \int_M \langle \nabla \times F, \nu \rangle \, ds.$$

Useful variations of Stokes.

(1) $f(x,w) = \sum_{j=1}^{\ell} \langle \theta_j(x), w \rangle v_j$, linear L-valued k-form.

$$\int_{\partial M} f(x, d\hat{x}_{\partial M}) = \sum_i \int_M \langle \nabla \wedge \theta_j, d\hat{x}_M \rangle v_j$$

$$= \sum_{i,j} \int_M \partial_{x_i} \langle \theta_j(x), e_i \lrcorner d\hat{x}_M \rangle v_j$$

$$= \int_M \sum_i \partial_{x_i} \left(\sum_j \langle \theta_j(x), e_i \lrcorner d\hat{x}_M \rangle v_j \right)$$

Def^{n}. Exterior derivative of 1-linear k-form $f(x,w)$ is the 1-linear k+1 form ..

$$f(x, \nabla \lrcorner w) := \sum_{i=1}^{n} \partial_{x_i} f(x, e_i \lrcorner w).$$

then,

$$= \int_M f(x, \nabla \lrcorner d\hat{x}_M).$$

(2). $M =$ n-dim body n-dim body in n-dim space X. (Assume Euclidean for simplicity).

∂M = $(n-1)$-dim hypersurface with outward pointing unit normal vector ν.

$$\int_{\partial M} f(x, \nu(x))\, dx = \int_{\partial M} f(x, e_{\bar{n}} \llcorner d\hat{x}_{\partial M}).$$

$$e_{\bar{n}} = \nu \wedge \frac{d\hat{x}_{\partial M}}{|d\hat{x}_{\partial M}|}.$$

linear 1-form

(abuse of notation, identify $*(\nu)(x) = d\hat{x}_{\partial M}$.

Define $g(x,w) := f(x, e_{\hat{n}} \llcorner w)$, n-form.

Apply Stokes:

$$= \int_M g(x, \nabla \lrcorner d\hat{x}_M).$$

$$= \int_M \sum_i \partial_{x_i} f(x, e_{\hat{n}} \llcorner (\underbrace{e_i \lrcorner d\hat{x}_M}_{= e_{\hat{n}}\, dx})).$$

$$= \int_M \sum_i \partial_{x_i} f(x, \ast(e_i \ast)) \; dx.$$

$$= \int_M f(x, \nabla) \, dx.$$

acts on this variable.

i.e., $\int_{\partial M} f(x, \nu(x))\, dx = \int_M f(x, \nabla)\, dx.$

<u>Example:</u> Gauss' div-th^m in $\mathbb{R}^n$.

$V(x)$ = vector field.

$f(x,\nu) = V(x) \cdot \nu(x).$

$$\Rightarrow \int_{\partial M} V(x) \cdot \nu(x)\, dx = \int_M \underbrace{V(x) \cdot \nabla}_{= \operatorname{div} V(x)}\, dx.$$