Lecture 9

08/10/2014.

(X, V) affine space

$\{e_i\}$ basis for V.

$\{e_i^*\}$ dual basis for V^*

Nabla symbol. $\nabla = \sum_{i=1}^n \underbrace{e_i^*}_{\text{act algebraically.}} \underbrace{\partial_{x_i}}_{\text{act by diff.}}$

<u>Examples:</u> Exterior derivative

$\theta : X \to \Lambda V^*$

flat affine space, so simply componentwise.

$$\underbrace{\nabla \wedge \theta}_{} = \sum_{i=1}^n e_i^* \wedge \overbrace{\partial_{x_i} \theta(x)}^{} . \qquad (d\theta)$$

Interior derivative:

$F : X \to \Lambda V$.

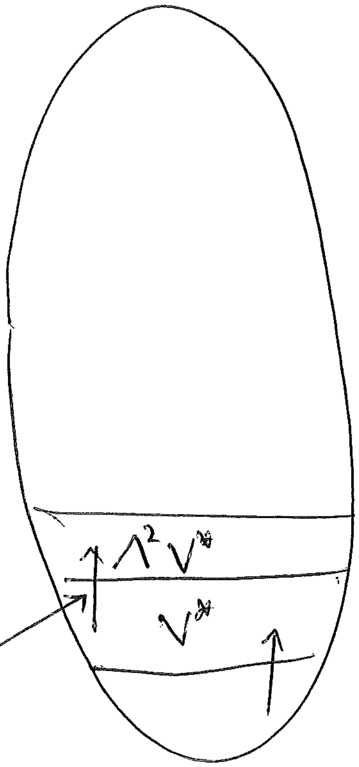
$$\nabla \lrcorner F(x) = \sum_{i=1}^n e_i^* \lrcorner \partial_{x_i} F(x) \qquad (\pm \delta F, \ d^* F)$$

<u>Note:</u> Any bilinear product $V^* \times L \to L'$ induces a nabla operator. like L-val. fun $\longmapsto$ L' val-func.

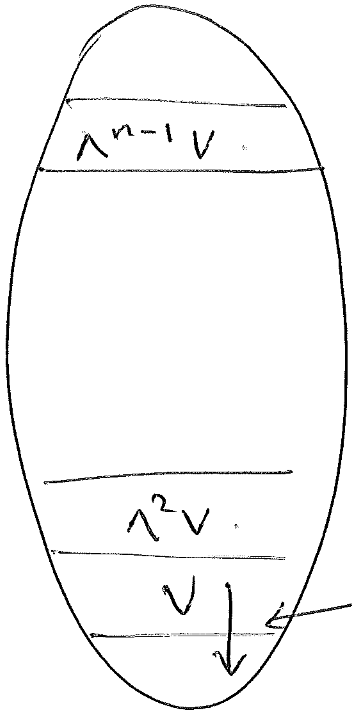
$$\begin{array}{ccc} V^* \times L & \longrightarrow & L' \\ (\text{lifts}) \downarrow & \nearrow & \\ L(V, L) \simeq V^* \otimes L & & \end{array}$$

Examples.

$\Lambda^* V$



ΛV.



$\delta F = \mathrm{div} \cdot F$.
$= \langle \nabla, F \rangle$
$= \nabla \cdot F$.

$d\theta = \mathrm{curl}\, \theta$

d acts here.

$d : \Lambda^k V^* \to \Lambda^{k+1} V^*$

δ acts here.

$\delta : \Lambda^k V^* \to \Lambda^{k-1} V^*$.

In 3-dim, identify this with V^* and with inner product, with V.

2 kinds of dualities b/w d and δ:

- L^2 duality.

$$\begin{aligned}
\int_X \langle d\theta, F \rangle \, dx &= \textstyle\sum_i \int_X \langle e_i^* \wedge \partial_{x_i}\theta, F \rangle \, dx. \\
&= \textstyle\sum_i \int_X \langle e_i \wedge \partial_{x_i}\theta, F \rangle \, dx. \\
&= \textstyle\sum_i \int_X \langle \partial_{x_i}\theta, -e_i \lrcorner F \rangle \, dx. \\
&= \textstyle\sum_i \int_X \langle \theta, -e_i \lrcorner \partial_{x_i} F \rangle \, dx \\
&= \int_X \langle \theta, \delta F \rangle.
\end{aligned}$$

Andreas does not want hidden -ves.

$\Rightarrow d^* = -\delta$.

• Hodge duality:

$$\theta : X \to \Lambda^k V^*, \quad \text{then} \quad \theta * : X \to \Lambda^{n-k} V.$$

Then,

$$\delta(\theta *) = \nabla \lrcorner (\theta *) = \nabla \lrcorner (\theta \lrcorner e_{\bar{n}})$$

$$= (\underset{\text{k-vector}}{\theta} \wedge \underset{\text{1-covector op.}}{\nabla}) \lrcorner e_{\bar{n}}$$

$$= (-1)^k (\nabla \wedge \theta) \lrcorner e_n.$$

$$= (-1)^k \, d\theta \lrcorner e_n.$$

$$= (-1)^k (d\theta) *.$$

$$\delta(\theta *) = (-1)^k (d\theta) *.$$

<u>Properties of d:</u>

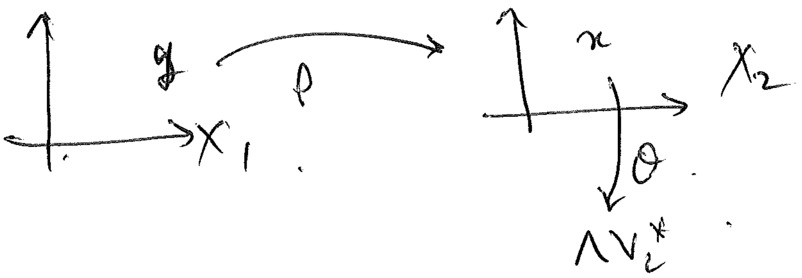
• Nilpotence: $d^2 = 0$. $\quad \nabla \wedge (\nabla \wedge \theta) = (\nabla \wedge \nabla) \wedge \theta = 0.$

<u>Example:</u> $\operatorname{curl}(\operatorname{grad} \theta) = 0$, $\operatorname{div}(\operatorname{curl} \theta) = 0$ in 3D.

• Invariance of d under pullback:

$\varphi : V_1 \to V_2$, smooth, but possibly injective.

$\theta : V_2 \to \Lambda V_2^*$. (really $X_2 \to \Lambda V_2^*$).



Consider the case.

(1) θ = scalar function, i.e. $\theta \in \Lambda^0 V_2^*$.

$$\underbrace{\nabla_y \theta(\rho(y))}_{\nabla_y \wedge \theta \text{ since work on functions.}} = \sum_k e_k^* \frac{\partial x_j}{\partial y_k} \frac{\partial \theta}{\partial x_j}$$

$$\underline{P}_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}.$$

$$\underline{P}_y^t((e_j')^*) = \sum_k \frac{\partial x_j}{\partial y_k} e_k^*.$$

$$= \sum P_y^t((e_j')^*) \frac{\partial \theta}{\partial x_j}$$

$$= \underline{P}_y^t(\nabla_x \theta(\rho(y))).$$

<u>Def</u>ⁿ. The pullback of $\theta: V_2 \to \Lambda V_2^*$ is

$$(\rho^* \theta)(y) = \underline{P}_y^t(\theta(\rho(y))).$$

lifted ~~pull~~ map via universal property, not just the scalar.

<u>Th</u>^m. $d\rho^* = \rho^* d$.

<u>Pf</u>. (1) Scalar $\to$ already checked.

(2). covector field $\theta = \sum \theta_i e_i^*$.

$$\nabla_x \wedge \theta = \sum_{ij} \frac{\partial \theta_i}{\partial x_j} (e_j')^* \wedge (e_i')^*$$

$$\rho^*(\nabla_x \wedge \theta) = \sum_{ij} \frac{\partial \theta_i}{\partial x_j} \rho_y^t (e_j)^* \wedge (e_i)^*.$$

$$= \sum_{ijkl} \frac{\partial \theta_i}{\partial x_j} \frac{\partial x_j}{\partial y_k} \frac{\partial x_i}{\partial y_l} e_k^* \wedge e_l^*.$$

Now check the LHS:

$$\rho^* \theta = \sum \theta_i \frac{\partial x_i}{\partial y_j} e_j^*$$

$$\nabla_y \wedge (\rho^* \theta) = \sum_{ijk} e_k^* \wedge \Big(\Big\{ \sum_l \frac{\partial x_l}{\partial y_k} \frac{\partial \theta_i}{\partial x_l} \Big\} \frac{\partial x_i}{\partial y_j} \text{ er}$$

$$+ \theta_i \frac{\partial^2 x_i}{\partial y_k \partial y_j} \Big) e_j^*.$$

By permuting indices $(j \leftrightarrow l)$, find first part the same. Need:

$$\sum_{k,j} \frac{\partial^2 x_i}{\partial y_k \partial y_j} e_k^* \wedge e_j^* = 0.$$

commute anti-commute.

(3) Multi-covector field.

By linearity, suffices to consider $\theta = \theta_1 \wedge \dots \wedge \theta_k$.

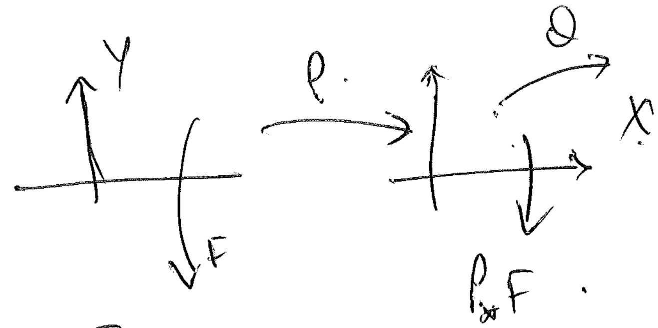
$$d\theta = \sum_{ij} e_i^* \wedge \theta_1 \wedge \dots \wedge \partial_i \theta_j \wedge \dots \wedge \theta_k$$

$$\rho^*(d\theta) = \sum_{ij} (\rho^* e_i^*) \wedge (\rho^* \theta_1) \wedge \dots \wedge (\rho^* \partial_{x_i} \theta_j) \wedge \dots \wedge (\rho^* \theta_k)$$

$$d(\rho^* \theta) = \sum_{ij} e_i^* \wedge \rho^* \theta_1 \wedge \dots \wedge \partial_{y_i} (\rho^* \rho^* \theta_j) \wedge \dots \wedge \rho^* \theta_k.$$

Need: $\sum (\rho^*(e_i'))^* \wedge \rho^* \partial_{x_i} \theta_i = \sum e_i^* \wedge \partial_{y_i}(\rho^*\theta_i)$, which is true by (2). $\blacksquare$

<u>Properties for δ:</u>



- Nilpotence : $\delta^2 = 0$.

$$\nabla \lrcorner (\nabla \lrcorner F) = \underbrace{(\nabla \wedge \nabla)}_{\substack{\| \\ 0}} \lrcorner F.$$

- Invariance of δ under pushforward <u>(normalised)</u>.

Pointwise duality:

$$\langle \rho_y^\vee \theta(y), F(y) \rangle_y = \langle \theta \, \underline{\rho_y^t}\, \theta(y), F(y) \rangle_y$$

$$= \langle \theta(x), \underline{\rho_y} F(y) \rangle_y \qquad \underline{\underline{x = \rho^{-1}(y)}}.$$

<u>Def^n</u> Push forward.

$$(\rho_* F)(x) = \rho_{\rho^{-1}(x)} F(\rho^{-1}(x)).$$

$\uparrow$

<u>need</u>.

- L^2-duality:

$$\int_{V_1} \langle (\rho^* \theta)(y), F(y) \rangle \, dy = \int_{V_1} \langle \underline{\rho_y^t}(\theta(\rho(y))), F(y) \rangle \, dy.$$

$$= \int_V \langle \theta(x), P_{\rho^{-1}(x)} F(\rho^{-1}(x)) \rangle \frac{1}{J_\rho(y)} \, dx .$$

$$J_\rho(y) = \left| \frac{dx}{dy} \right| = \left| \underline{\rho} \big|_{\Lambda^n V_1} \right| :$$

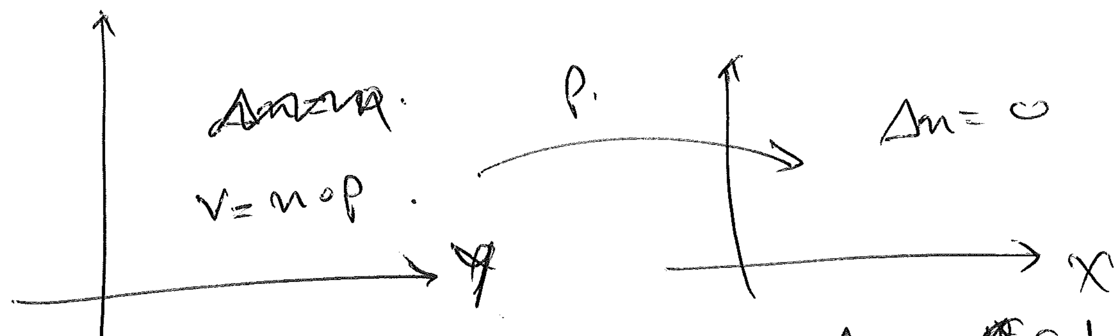
Jacobian.

Def^{n}. Normalised pushforward.

$$(\tilde{P}_* F)(x) := \frac{1}{|J_\rho(\rho^{-1}(x))|} (P_* F)(x).$$

Th^{m}. $\delta \tilde{P}_* = \tilde{P}_* \delta$ for $\tilde{P}_*$ normalised pushforward.

Example.



$\Delta n = \cancel{d}\, \delta dn.$

what does v solve?

$$v = \rho^* n \Rightarrow n = \rho^{*-1} v = (\rho^{-1})^* v .$$

$0 = \Delta n = \delta \rho^{*-1}(dv)$, multiply by $\tilde{P}_*^{-1}$.

$$0 := \tilde{\rho}_*^{-1} \, \delta \, \rho^{*-1} (dv).$$

$$= \delta \left(\underbrace{\tilde{\rho}_*^{-1} \rho^{*-1}} \right) dv.$$

$$J_\rho \, (\rho^* \rho_*)^{-1}; \quad \text{but} \quad \cancel{\langle \rho^* (\rho_* x), v \rangle}$$

$$\langle \rho^* \rho_* x, v \rangle = \langle \rho_* x, \rho_* v \rangle =: g(x, v)$$

metric

So, $0 = \operatorname{div} \left(\sqrt{\det g} \; g^{-1} \right) \nabla v.$