Lecture 8

24/09/2014.

## Construction of $\Lambda V$.

The key result:

**Th$^{m}$.** All automorphism $\varphi: \mathcal{L}(\mathbb{C}^n) \to \mathcal{L}(\mathbb{C}^n)$ are inner.

I.e., $\exists\, T \in \mathcal{L}(\mathbb{C}^n)$ s.t. $\varphi(x) = TxT^{-1}$, $x \in \mathcal{L}(S)$.

**Pf.** Consider minimal left ideals: these are parametrized by $v \in S$, $v \neq 0$. I.e.,

$$J_v^{l} = \{ x v^t ;\ x \in \mathbb{C}^n \}. \quad ( N(xv^t) \supset [v] ).$$

Fix $v_1$. Then $\varphi(J^l_{v_1})$ must also be a min. left ideal.

So, $\exists \tilde{v}_1$: $\varphi(xv_1^t) = y\tilde{v}_1^t$, map $T: x \mapsto y$ lin.

So $\varphi(xv_1^t) = Tx\tilde{v}_1^t$. Also, this map is invertible b/c $\varphi$ aut.

Minimal right ideals

$$J_v^r = \{ v y^t :\ y \in \mathbb{C}^n \}.$$

$\forall v_2 \exists \tilde{v}_2 \in \mathbb{C}^n$, $T_2 : \mathbb{C}^n \to \mathbb{C}^n$

$$\varphi(v_2 y^t) = \tilde{v}_2 (T_2 y)^t.$$

Choose $v_1, v_2$: $\langle v_1, v_2 \rangle = 1$. $(= v_1^t v_2)$.

$$\varphi(xv_1^t v_2 y^t) = \varphi(xv_1^t)\varphi(v_2 y^t) = (T_1 x \tilde{v}_1^t)(\tilde{v}_2 T_2 y)^t = T_1 x y^t T_2.$$

$\Rightarrow \varphi(x) = T_1 x T_2^t$ on select $x = I$, $I = \varphi(I) = T_1 T_2^t$

$\Rightarrow T_1 = T_2^{t\,-1}$.

$V$ real inner product space.

$\rho_i : V \to \mathcal{L}_{\mathbb{C}}(S_i)$, $i = 1, 2$, rep. $\rho_i^2(v) = \langle v, v\rangle I_S$.

Let such $\rho_i$ be given with $\dim S_i = 2^{\lfloor n/2 \rfloor}$.

$\iff$ $\rho_i : \Delta V_{\mathbb{C}} \to \mathcal{L}(S_i)$.

<u>$n$ = even</u>: $\rho_i : \Delta V_{\mathbb{C}} \to \mathcal{L}(S_i)$. are invertible by lec. 7.

Fix $\tilde{T} : S_1 \to S_2$ invertible. $\rho_2 \rho_1^{-1} : \mathcal{L}(S_1) \to \mathcal{L}(S_2)$,
alg. isom.

alg. $\xrightarrow{}$ iso

Consider $\tilde{T}^{-1}(\rho_2 \rho_1^{-1}(x))\tilde{T}$, an automorphism on $\mathcal{L}(S_1)$.

By Th$^{m}$, $\exists\, T_0 \in \mathcal{L}(S_1)$ invertible s.t.

$$\tilde{T}^{-1}(\rho_2 \underbrace{\rho_1^{-1}(x)}_{w})\tilde{T} = T_0 x T_0^{-1} \quad \forall x \in \mathcal{L}(S_1).$$

So, $\rho_2(w) = \underbrace{\tilde{T} T_0}_{T} (\rho_1(w)) \underbrace{(\tilde{T} T_0)^{-1}}_{T^{-1}}$.

$\exists\, T \in \mathcal{L}(S_1, S_2)$ s.t. $\forall w \in \Delta V_{\mathbb{C}}$, $\forall z \in S_1$,

$$\rho_2(w) T z = T(\rho_1(w) . z)$$

$$\begin{array}{ccc} & \overset{\rho(w)}{\curvearrowright} & S_2 \\ w & & \\ \Delta V_{\mathbb{C}} & & \uparrow T \\ & \underset{\rho_1(w)}{\curvearrowright} & S_1 \end{array}$$

Note: This map $T$ is unique upto scalars.

n odd: Consider restrictions to the even subalgebra.

$\rho_i : \Delta^{ev} V_c \to \mathcal{L}(S_i)$. are isomorphisms. (thm lec. 7).

Similarly to above: $\exists\, T \in \mathcal{L}(S_1, S_2)$: $\rho_2(w) \circ T = T \circ \rho_1(w)$ holds for all $w \in \Delta^{ev} V_c$.

The role of $\Delta^n V_c$. (n-vectors).

Fix one $w_{\bar{n}} \in \Delta^n V_c$ s.t. $w_{\bar{n}}^2 = +1$.

n even: ~~Fix~~ $\rho(w_{\bar{n}}) \in \mathcal{L}(S)$. has two eigenvalues.

$\sigma(\rho(w_{\bar{n}})) = \{\pm 1\}$, $S = S_+ \oplus S_-$ $S_\pm$ eigenspaces for $\pm 1$.

vector $\Delta^1 V$ swap $S_+$ & $S_-$.

n odd: $\rho(w_{\bar{n}})$. as before; but $\rho$ is not an isomorphism.

So, either $\rho(w_{\bar{n}}) = \varepsilon_\rho I$ $\varepsilon_\rho = +1$ or $-1$.

Claim: For two reps, (I) $\varepsilon_{\rho_1} = \varepsilon_{\rho_2}$ or (II) $\varepsilon_{\rho_1} = -\varepsilon_{\rho_1}$.

Then, for each case:

(I). $\rho_2(w) \circ T = T \circ \rho_1(w)$ $\forall w \in \Delta V_c$.

(II). $\rho_2(\hat{w}) \circ T = T \circ \rho_1(w)$. $\forall w \in \Delta V_c$.

Pf. $w = w_1 + w_{\bar n} \wedge w_2$ , $w_i \in \Lambda^{ev} V_c$ , since $w$ odd.

$\rho(w) = \rho(w_1) + \rho(w_{\bar n})\rho(w_2) = \cancel{\rho(w_1)} \pm \rho(w_1 \mp w_2)$.

Def$^{n}$. $V$ = real inner product space of dim. $n$.

Fix one complex representation.

$$\rho : V \longrightarrow \mathcal{L}_{\mathbb{C}}(S)$$

of minimal dimension $2^{\lfloor n/2 \rfloor}$. Write $S = \not{\Delta} V$, and call this the complex spinor space.

Notation: $\underset{\not{\Delta}V}{\overset{\uparrow}{w}} \cdot \underset{\not{\Delta}}{\overset{\uparrow}{z}} := \rho(w) z$.

$n$ even: $\not{\Delta} V = \not{\Delta}^+ V \oplus \not{\Delta}^- V$. ( eigenspaces $\rho(w_{\bar n})$, $\rho(w_{\bar n})^2 = +1$ ).

$n$ odd:

Write $\begin{cases} \rho_+(w) = \rho(w). \\ \rho_-(w) = \rho(\hat{w}). \end{cases}$

Note: • $\rho$ above is "unique upto isomorphism" ($n$ even).

• these isomorphisms are "unique up to scalars"

$\rightarrow$ get rid of scalars & simplify the case $\pm 1$.

# Induced mappings of spinors.

$V_1, V_2$ inner prod spaces.

$T: V_1 \to V_2$ invertible linear map (not. nec. isometry).

(a) for $\Lambda V \Rightarrow$. $T_\wedge : \Lambda V_1 \to \Lambda V_2$ = $\wedge$-~~homo~~ iso morphism.

$T_\wedge(v_1 \wedge \dots \wedge v_n) = (Tv_1) \wedge \dots \wedge (Tv_n)$.

<u>Claim</u>: $T_\wedge : \Delta V_1 \to \Delta V_2$ is a $\Delta$-isomorphism.

$\Longleftrightarrow$ $T$ is an isometry.

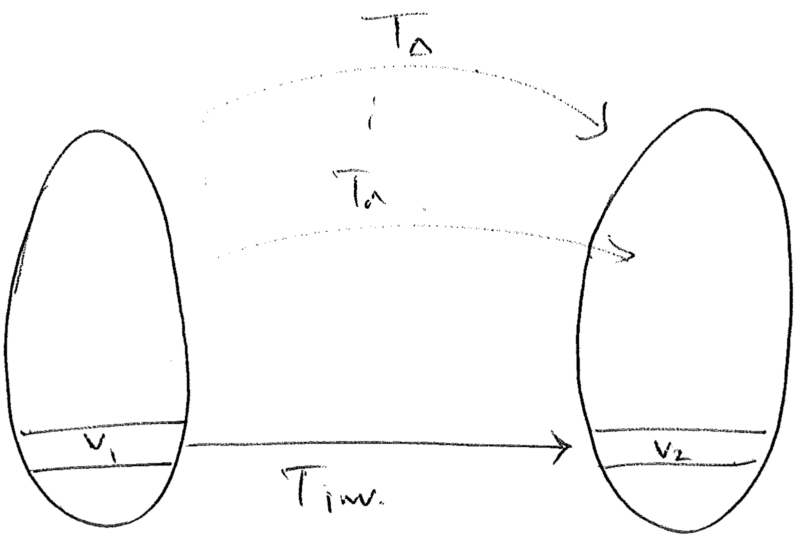
<u>Pf</u> ($\Rightarrow$). $|Tv|^2 = (Tv)^2 = T(v^2) = T(|v|^2 1) = |v|^2$.

($\Leftarrow$). map $V_1 \to \Delta V_2$. $v \mapsto Tv$. (since $\Delta^1 V_1 = V_1$).

Check (C). $(Tv)^2 = |Tv|^2 \overset{?}{=} |v|^2$. (by assumption).

(AC). for $\Delta V_1 \Rightarrow$. $\tilde{T} : \Delta V_1 \to \Delta V_2$, we see that.

$\tilde{T} = T_\wedge$.



If $T$ is <u>not</u> an isometry, how do we construct such a $\Delta$-isomorphism $T_\Delta$? (<u>Note</u>: This is not the extension of $T_\wedge$).

<u>Answer</u>: Polar decomposition

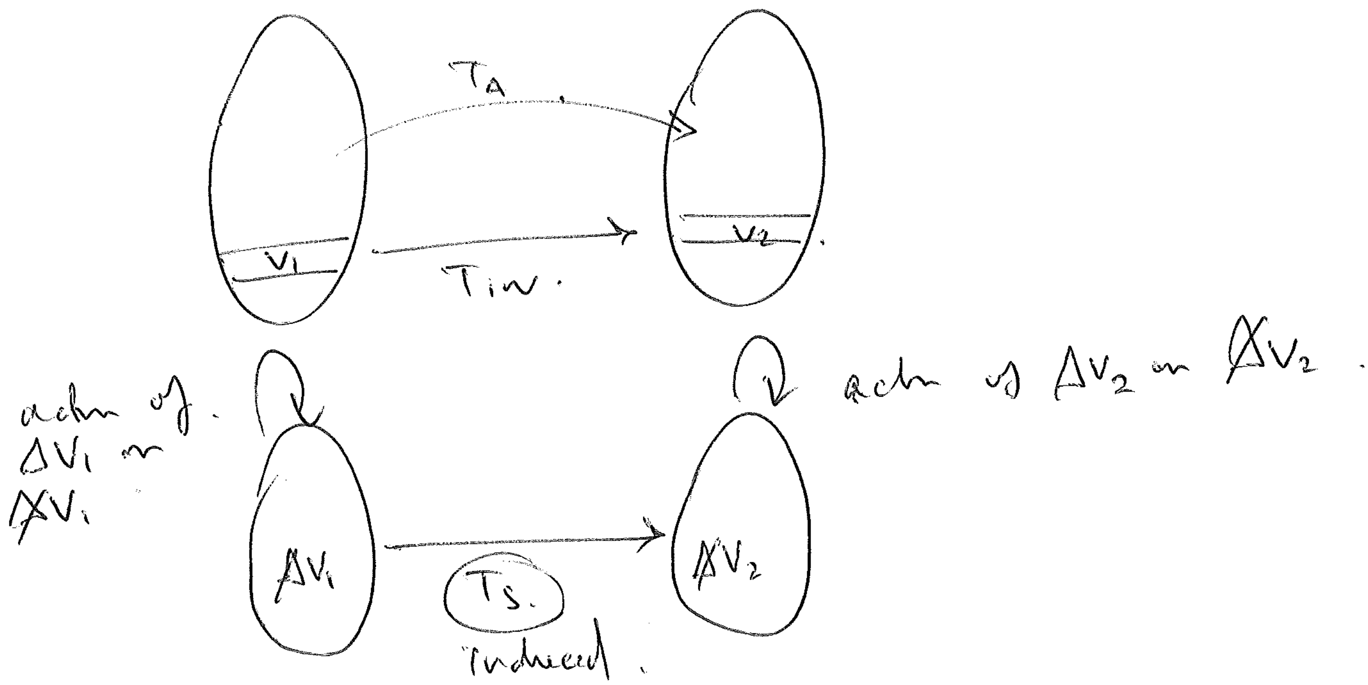
$$T = S_2 \mathcal{U} = \underbrace{\mathcal{U}}_{\substack{\text{isometry} \\ V_1 \to V_2}} S_1 .$$

$V_1 \to V_2$. $\uparrow$ Sym map in $V_2$ $\qquad$ $\uparrow$ sym map in $V_1$.

$$\boxed{\begin{aligned} \mathcal{U} &= T(T^*T)^{-\frac{1}{2}} \\ &= (TT^*)^{-\frac{1}{2}} T. \end{aligned}}$$

$\mathcal{U}$ is the polar isometric factor of $T$.

For $T \xrightarrow{\text{get to}} \mathcal{U}$ and obtain $T_\Delta = \mathcal{U}_\wedge$.




Obtain an induced map $T_s : \Delta V_1 \to \Delta V_2$.

that satisfies

$$T_s(w\cdot\psi) = \mathcal{U}(w)\cdot(T_s\psi). \quad \forall w\in\Delta V_1,\ \psi\in\Delta V_1 \quad \left(\text{and } w\in\Delta^{ev}V_1 \text{ if } \dim V_1 = \text{odd}\right).$$

<u>Note:</u> $T_s$ is unique upto scalars. Ie, $\lambda T_s$ are all possible such maps, $\lambda\in\mathbb{C}$.

To make $T_s$ unique only upto $\pm 1$, we require that $T_s$ further satisfy:

(I). $\langle T_s\psi, T_s\psi\rangle_{*,2} = \langle\psi,\psi\rangle_{*,1}$

(II). $(T_s\psi)^{\dagger_2} = T_s(\psi^{\dagger_1})$

$\dagger_i$ - spinor conjugate. $\dagger$ \dagger, physics notation.

$\langle\cdot,\cdot\rangle_{*,i}$ - spinor dualities. (more gen than inner prod).

<u>Requirements.</u> Fix $V = V_1$, $\underset{\substack{\uparrow\\ \text{sesquilinear.}}}{\langle\cdot,\cdot\rangle_*} = \langle\cdot,\cdot\rangle_{*,1}$

(I) $\underset{\substack{\uparrow\\ \text{complex}}}{\forall w\in\Delta V}$. $\langle w\cdot\psi, \varphi\rangle_* = \langle\psi, \bar{w}^c\varphi\rangle_*$ ← such a duality always exists, unique upto scalars.

conj each comp.

(II). $(w\cdot\psi)^\dagger = w^c(\psi^\dagger)$

$\exists$ such $\dagger$, unique upto scalars.