Lecture 7

23/09/2014.

Complex representations of ΔV

V = real inner product space.

ΔV = real Clifford algebra $:= \{ \sum_{S \subset \bar{n}} a_S e_S : a_S \in \mathbb{R} \}$.

ΔV_c = complexified of ΔV.

if

$= \{ w_1 + i w_2 : w_i \in \Delta V \} = \{ \sum_{S \subset \bar{n}} a_S e_S : a_S \in \mathbb{C} \}$.

($\Delta(V_c) = (\Delta V)_c$).

Def^n S = complex linear space

$\rho : V \longrightarrow \mathcal{L}(S)$. real linear for V real space.

(by forgetting $\mathbb{C}$ action on $L(S)$ and just restriction to $\mathbb{R}$ action)

is a complex representation of V on S if

$\rho^2(v) = \langle v, v \rangle I_S$

(I) Such a ρ induces a complex algebra homom.

$\rho : \Delta V_c \longrightarrow \mathcal{L}(S)$.

$$\begin{tikzcd}
\Delta V_c \arrow[r, dashed, "\rho"] & L(S). \\
V \arrow[ur, "\rho"'] &
\end{tikzcd}$$

① First regard $L(S)$ as a real linear space

then (U) for $\Delta V \Rightarrow \rho$ extends to a real homom.

$$\rho : \Delta V \to \mathcal{L}(S)$$

Now regard $L(S)$ complex and extend ρ by linearity:

$$\rho(w_1 + i w_2) = \rho(w_1) + i\rho(w_2).$$

(II). Conversely, and complex homom. $\rho : (\Delta V)_c \to \mathcal{L}(S)$. restricts to $\rho : V \to \mathcal{L}(S)$.

$$\rho(v)^2 = \rho(v^2) = \rho(\langle v, v\rangle 1) = \langle v, v\rangle I_S.$$

(III). For any representation $\rho : V \to \mathcal{L}(S)$ with $\dim_{\mathbb{R}} V = n = 2k$ (or $2k+1$), we have

$$\dim_{\mathbb{C}} \geq 2^k.$$

Pf $\rho(e_S) = \rho(e_{s_1}) \cdots \rho(e_{[illegible]})$.

$$\sum_{S \subset \bar{n}} a_S \rho(e_S) = 0.$$

$$\rho(e_i) \sum_{S \subset \bar{n}} a_S \rho(e_S) \rho(e_i)^{-1} = 0.$$

$$\Rightarrow \quad a_S \rho(e_S) + a_{\bar{n} \setminus S} \rho(e_{\bar{n} \setminus S}) = 0.$$

n odd: $\dim L(S) \geq 2^{n-1} = 2^{2k}$ $\Rightarrow$ $\dim S \geq 2^k$. (by square roots).

__n even:__ Conjugate with $\rho(e_i) \Rightarrow a_s = 0 \quad \forall s$.

$\left(\text{decouples, ie, when } s = (1,2),\ n \setminus s = (3,4)\right) \Rightarrow \dim \mathcal{H}(S) \geq 2^n \geq 2^{2k}$.

__Goal:__ prove existenceness and uniqueness of representations of minimal dimension 2^k.

The Standard representation.

Fix ON-basis for V indexed

$$e_{-k},\ e_{-(k-1)}, \dots, e_{-1},\ e_0,\ e_1, \dots, e_k.$$

$\uparrow$ present if n odd, otherwise, forget it.

$\lfloor \frac{n}{2} \rfloor = k$.

$\mathcal{K} := \{-k, \dots, -1, 0, 1, \dots, k\}$ for k odd.

$:= \{-k, \dots, -1, +1, \dots, k\}$ for k even.

1. Euclidean case:

$V_0 := \operatorname{span}\{e_1, \dots, e_k\} \subset V$.

$S := \Lambda_{\mathbb{C}} V_0$ (allow for complex coeff.).

$\psi \in S$, $\rho(e_j)\psi := e_j \psi = e_j \lrcorner \psi + e_j \wedge \psi$.

Extend ρ to e_{-j}: $\rho(e_{-j})\psi := i(-e_j \lrcorner \psi + e_j \wedge \psi)$

$\nearrow$ so that $\rho^2 = +1$.

For odd dim, $\rho(e_0)\psi := \hat{\psi}$ $\nwarrow$ reverse sign on Λ^{odd} part of $\Lambda_{\mathbb{C}} V_0$.

Extend by linearity to ~~ρ~~ $\rho : V \longrightarrow \mathcal{L}(S)$.

<u>Claim:</u> ρ is a representation and $\dim S$ is minimal!.

2. Non-Euclidean case.

If $e_j^2 = -1$ in ΔV, then just multiply $\rho(e_j)$ as we defined for Euclidean case by i.

<u>Example</u> $V =$ Euclidean 3space $= \mathbb{R}^3$, $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ std basis

$$\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \quad \tilde{e}_1 = e_1, \quad \tilde{e}_2 = e_{-1}, \quad \tilde{e}_3 = e_0 .$$

$$\underset{\Lambda[e_1]}{\overset{\|}{S}} - \text{basis} \quad \{1, e_1\}.$$

$[e_1]$: line spaned by e_1.

$$\rho(\tilde{e}_1) = \rho(e_1) = \begin{array}{cc} \swarrow e_1 \Delta_+ 1 & \swarrow e_1 \Delta_+ e_1 \\ \end{array}$$

$$\rho(\tilde{e}_1) = \rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [e_1 \ \ 1]$$

$$\rho(\tilde{e}_2) = \rho(e_{-1}) = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$$\rho(\tilde{e}_3) = \rho(e_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are the classical Pauli-matrices for the non-relatavistic formulation for the electron.

Example: V = physical spacetime

$$\begin{array}{cccc} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \\ - & + & + & + \\ & | & | & | \\ -ie_1 & e_{-2} & -e_2 & e_{-1} \end{array}$$

(other convention is $+ - - -$, but then, rep is $\mathbb{H}^2$).

Rep in $\mathbb{R}(4)$.

$\nearrow$ squares positive

(Cheating so we arrive at right matrices)

S-basis : $\{1, e_{12}, e_1, e_2\}$.

$$\rho(\tilde{e}_0) = -i\rho(e_1) = -i\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$e_1\Delta_+$.

$$\rho(\tilde{e}_1) = \rho(e_{-2}) = i\left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

$e_2\Delta_-$:

$$\rho(\tilde{e}_2) = -\rho(e_2) = -\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right].$$

$e_2\Delta_+$

$$\rho(\tilde{e}_3) = \rho(e_{-1}) = i\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right).$$

$e_1\Delta_-$:

Note: 2×2 blocks off diag, like Pauli-matrices.

$$\{\tilde{e}_0; \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \longleftrightarrow \{-i\gamma_5, -i\gamma_1, -i\gamma_2, i\gamma_3\}.$$

Uniqueness upto isomorphism of S.

Th^m. Consider $\mathcal{L}(\mathbb{C}^n)$, and let $\Psi : \mathcal{L}(\mathbb{C}^n) \to \mathcal{L}(\mathbb{C}^n)$ be an automorphism.

Then, $\exists\, T \in \mathcal{L}(\mathbb{C}^n)$ s.t.

$$\Psi(x) = T x T^{-1} \quad \forall x \in \mathcal{L}(\mathbb{C}^n).$$

Lemma

(I) All left ideals in $\mathcal{L}(\mathbb{C}^n)$ are of the form

$$\overset{\text{left}\to}{\mathcal{J}}{}^{l}_{\underset{\underset{\subseteq_{\text{sub}}\mathbb{C}^n}{\nearrow}}{\mathcal{U}}} := \{T \in \mathcal{L}(\mathbb{C}^n) : N(T) \supset \mathcal{U}\}.$$

(II) All right ideals. in $\mathcal{L}(\mathbb{C}^n)$ are of the form

$$\mathcal{J}^{r}_{\mathcal{U}} := \{T \in \mathcal{L}(\mathbb{C}^n) : \operatorname{Ran}(T) \subset \mathcal{U}\}.$$

(III). Two sided ideals are only $\{0\}$ and $\mathcal{L}(\mathbb{C}^n)$.

I.e., for each subspace $\mathcal{U} \subset \mathbb{C}^n$, define $\mathcal{J}^{l}_{\mathcal{U}}$ and $\mathcal{J}^{r}_{\mathcal{U}}$. The statement is that every ideal arises in this way. I.e., $\exists$ such a subspace.

Pf. Given $\mathcal{J}$ = left ideal in $\mathcal{L}(\mathbb{C}^n)$.

Define $\mathcal{U} := \bigcap_{S \in \mathcal{J}} N(S)$.

Need $\mathcal{J} = \mathcal{J}^l_{\mathcal{U}}$.

$\mathcal{J} \subset \mathcal{J}^l_{\mathcal{U}}$: $T \in \mathcal{J}$; $N(T) \subset \mathcal{U} \Rightarrow T \in \mathcal{J}^l_{\mathcal{U}}$.

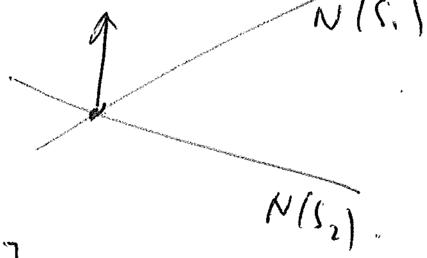
Now, $\mathcal{J}^l_{\mathcal{U}} \subset \mathcal{J}$: assume $T \in \mathcal{J}^l_{\mathcal{U}}$, ie $N(T) \subset \mathcal{U} = \bigcap_{S \in \mathcal{J}} N(S)$.

$Tx \neq 0 \Rightarrow \exists S \in \mathcal{J} : Sx \neq 0$.

If $S_1, S_2 \in \mathcal{J}$, then $\exists S \in \mathcal{J}$, s.t. $N(S) = N(S_1) \cap N(S_2)$.

$\Rightarrow \exists S_0^* \in \mathcal{J}$ s.t. $N(S_0^*) = \mathcal{U}$ by iteration. ~~(Have not only [illegible]~~.

Choose basis.



$$S_1 = \begin{bmatrix} | & | & | & | \\ 0 & 0 & x & y \\ | & | & | & | \end{bmatrix}.$$

$$S_2 = \begin{bmatrix} | & | & | & | \\ 0 & x' & 0 & y' \\ | & | & | & | \end{bmatrix}.$$

$N(S_1) \cap N(S_2)$. $N(S_1) \ominus N(S_2)$ $N(S_2) \ominus N(S_1)$ $(N(S_1) + N(S_2))^\perp$.

By left mult:
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \ \& \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Sum then
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 2I \end{pmatrix} \in J \text{ wh}.$$
$N(-) = N(S_1) \cap N(S_2)$.

Split $\mathbb{C}^n = N(S_0) \oplus N(S_0)^\perp$, $S_0 = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$, w.l.o.g.,

$$S_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

$N(T) \supset N(S_0)$.

$$\Rightarrow \quad T = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}}_{\in \mathcal{J}}}_{\in \mathcal{J}.}.$$

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