Lecture 6 22/09/2014.

Defⁿ V inner product space. An associative algebra $(A, +, \cdot, 1_A)$ is a <u>Clifford Algebra</u> if $\exists$ a linear map $\rho : V \to A$ satisfying

(c) $\rho(v)^2 = \langle v, v\rangle 1_A$.

(u) If $\rho' : V \to A'$ (another algebra $(A', +, \cdot, 1_{A'})$)

$\exists!\, T : A \to A'$ algebra homomorphism s.t.

$$\rho' = T \circ \rho$$

$$\begin{tikzcd} A \arrow[rr, dashed, "T"] & & A' \\ & V \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] & \end{tikzcd}$$

① $(\Lambda V, \Delta) = \Delta V$ is <u>a</u> Clifford Algebra.

(c) $v \Delta v = \langle v, v \rangle$. by construction.

Note Polarising (c), we find that (c) is equivalent to: $\rho(v_1)\rho(v_2) + \rho(v_2)\rho(v_1) = 2\langle v_1, v_2\rangle 1_A$.

~~(u) suffices to show~~

(u) know that $\{e_S\}$ form a basis for $\Delta V = A$.

Set $T(e_S) := \rho'(e_{s_1}) \cdots \rho'(e_{s_k})$. is the homomorphism.

(2) Any two Clifford alg for V are canonically isomorphic.

$$\begin{tikzcd} A \arrow[rr, shift left, "T"] & & A' \arrow[ll, shift left, "T'"] \\ & V \arrow[ul, "\rho"] \arrow[ur, "\rho'"'] & \end{tikzcd}$$

~~why is $T'T = I$?~~

why is $T'T = I$? Consider

$$\begin{tikzcd} A \arrow[rr, "\delta"] & & A \\ & V \arrow[ul, "\rho"] \arrow[ur, "T'T\rho"'] & \end{tikzcd}$$

By uniqueness,

$\delta \circ \rho = T'T\rho$ $\qquad$

$= T'\rho'$ $\qquad \rho' = T\rho$

$\cancel{T\rho = \rho'} = \rho$ $\qquad \rho = T'\rho'$

~~Te~~, $\delta = I$. $\quad$ ~~$\delta o \rho =$~~

(3) (U) ~~for Cliff alg~~ is "almost automatic ("from (C))".

V $\dim V = n = n_+ + n_-$, sign $= \sigma = n_+ - n_-$

$\rho : V \to A$. Assume $\rho(V)$ generates A. (by 2?).

<u>Prop</u>: If $n_+ - n_- \not\equiv 1 \pmod 4$, then (U) holds.

If $n_+ - n_- \equiv 1 \pmod 4$ then (U) holds iff

$w = \rho(e_1) \cdots \rho(e_n)$ is <u>not</u> scalar.

Pf. $\rho(e_s) = \rho(e_{s_1}) \cdots \rho(e_{s_k})$. Need to prove that $\{\rho(e_s)\}_{s\subset\bar{n}}$ is a basis for A.

$$\sum_{s\subset\bar{n}} a_s \rho(e_s) = 0 \rightarrow \rho(e_i) \sum_{s\subset\bar{n}} a_s \rho(e_s) \rho(e_i)^{-1} = 0$$

for ~~some~~ each i

$$\rho(e_i)\rho(e_s) = \pm \rho(e_s)\rho(e_i).$$

Fix signs $\varepsilon_1, \dots, \varepsilon_n$

$$\Rightarrow \sum_{\{s\subset\bar{n} :\ \rho(e_s)\rho(e_i) = \varepsilon_i \rho(e_s)\rho(e_s)\}} a_s \rho(e_s) = 0. \quad (2^n \text{ equations}).$$

$$\Rightarrow a_s \rho(e_s) + a_{\bar{n}\setminus s} \rho(e_{\bar{n}\setminus s}) = 0.$$

~~x even:~~ $a_\emptyset \underset{1}{\underset{\shortparallel}{\rho(e_\emptyset)}} + a_{\bar{n}} \rho(e_{\bar{n}}) = 0.$

n = even:

$$\rho(e_i)\rho(e_{\bar{n}}) = -\rho(e_n)\rho(e_i).$$

$$\Rightarrow a_\emptyset, a_{\bar{n}} = 0 \Rightarrow (n).$$

n = odd: $(n = 2k+1)$.

$$\rho(e_{\bar{n}})^2 = (-1)^{\frac{n(n-1)}{2}} \rho(e_{\bar{n}}) \cdot \rho(e_{\bar{n}})$$

$$= (-1)^{\frac{n(n-1)}{2}} (-1)^n$$

$$= (-1)^{k(2k+1)+n} \dots$$

$$= (-1)^{k+n} \dots$$

bad case: $(-1)^{k+n_-} = 1 \iff k+n_-$ even.

So, $2k+2n_- \equiv 0$ ~~or~~ $\pmod 4$.

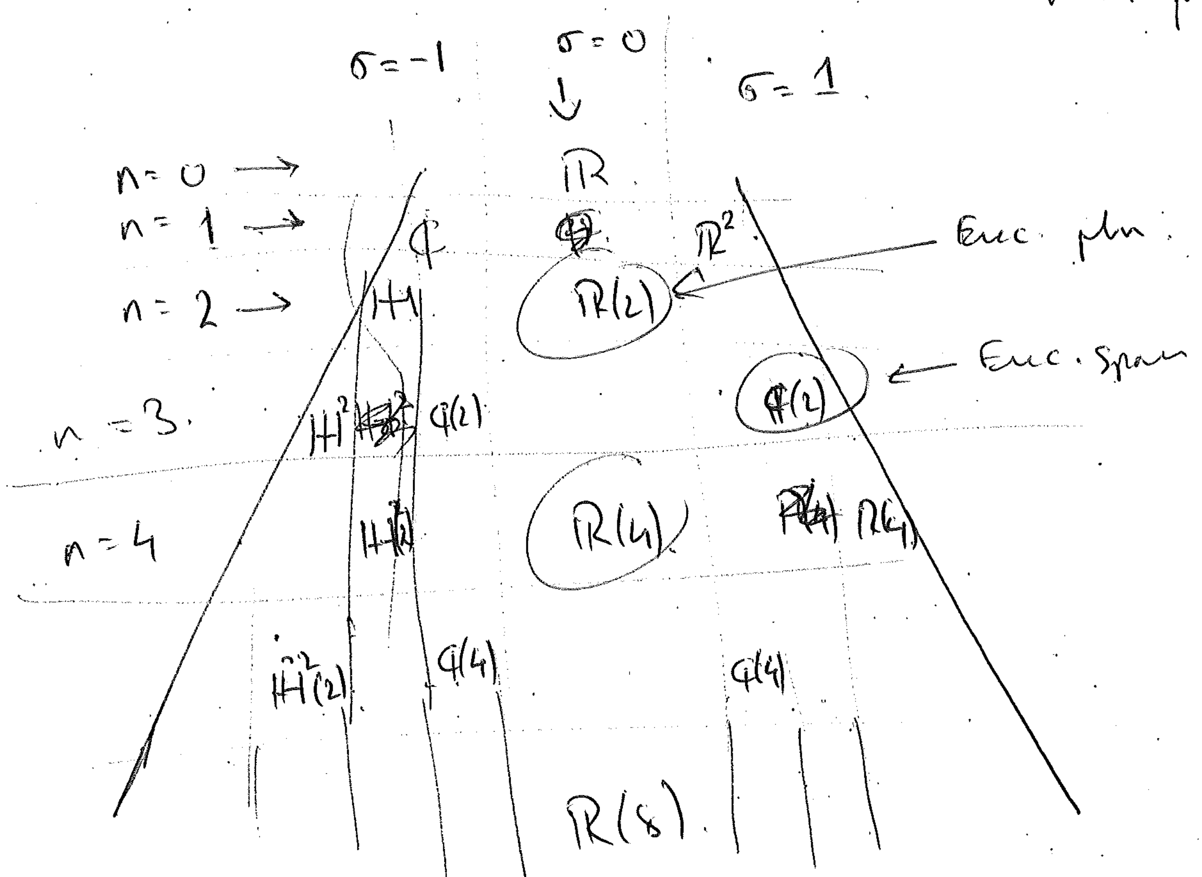
$$\begin{aligned} 2k+2n_- &\\ \| &\\ n-1+2n_- &= n_+ + n_- - 1 + 2n_- \\ &= n_+ \end{aligned}$$

$(-1)^{k-n_-} = (-1)^{k+n_-} \implies$ consider $2k - 2n_- \equiv 0$ $\pmod 4$ instead.

So, $n_+ + n_- - 1 - 2n_- = n_+ - n_- - 1 \equiv$ 0 or ~~2~~ $\pmod 4$.

So, $n_+ - n_- \equiv 1 \pmod 4$.

"Pascal's triangle of Clifford algebras" → computation next pages.



Classification of real Cliffords.

V $\quad n = n_+ + n_-$ dim.
$\quad \sigma = n_+ - n_-$ signature

$\Delta V \cong A(k)$, $\quad$ since $A(k) = \{ k \times k$ matrices with entries in an assoc. algebra $A \}$.

$A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{R}^2, \mathbb{H}^2$.

$\{(a,b)\}$, multiplication $(a_1, b_1)\cdot(a_2, b_2) = (a_1 a_2, b_1 b_2)$.

<u>Note:</u> no $\mathbb{C}^2$.

$(n, \sigma) = (0,0)$: $\quad \Delta\{0\} = \mathbb{R}$.

$(1,-1)$: $e_1^2 = -1$. $\quad \Delta V = \text{span}\{1, e_1\} \cong \mathbb{C}$

$(2,-2)$: $e_1^2 = e_2^2 = -1$. $\quad \Delta V = \text{span}\{1, e_1, e_2, e_{12}\} \cong \mathbb{H}$.

$(1,1)$: $e_1^2 = +1$. $\quad a + b e_1 \longleftrightarrow (a+b, a-b) \in \mathbb{R}^2$.

(Andreas says: $\mathbb{H}$ "anti-Euclidean space")

$(3,-3)$: $\quad e_1^2 = e_2^2 = e_3^2 = -1$, $\quad \omega = e_{123}$, $\omega_0^2 = +1$, $\omega_0 \in Z(\Delta V)$

centre, commutes with everything!

$\Delta V \ni w \longleftrightarrow ((1+e_{123})w, (1-e_{123})w)$.

Main case $\sigma = 0$, V_0 = Euclidean.

$$V = V_0 \oplus V_0 \quad , \quad \langle x + x' + y + y' \rangle_V := \langle x, y \rangle_{V_0} - \langle x', y' \rangle_{V_0}.$$

$$\rho : V \to \mathcal{L}(\Lambda V_0).$$

$$\rho(x+0)\, w := x \Delta w = x \lrcorner w + x \wedge w.$$

$$\rho(0+y)\, w := y \Delta w = -y \lrcorner w + y \wedge w.$$

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Check ρ satisfies (C):

$$\rho(e_i+0)\,\rho(0+e_j) + \rho(0+e_j)\,\rho(e_i+0) \overset{?}{=} 0.$$

$$\rho(e_i+0)\,\rho(0+e_j)\, w = e_i \lrcorner (-e_j \lrcorner w + e_j \wedge w) + e_i \wedge (-e_j \lrcorner w + e_j \wedge w).$$

$$\rho(0+e_j)\,\rho(e_i+0)\, w = -e_j \lrcorner (e_i \lrcorner w + e_i \wedge w) + e_j \wedge (e_i \lrcorner w + e_i \wedge w).$$

Anti-comm. relation repeatedly gives

So, $\rho : \underset{n \text{ dim}}{V} \longrightarrow \underset{2^{n/2} \text{ dim}}{\mathcal{L}(\Lambda V_0)}$.

$$\boxed{\text{In general: } n \longrightarrow n+2, \text{ same } \sigma.}$$

V_0 ~~basis~~ $\{\underbrace{e_1, \dots, e_n}_{V_0}, e_+, e_-\}$, $e_+^2 = 1$, $e_-^2 = -1$.

Define $\rho : V \to (\Lambda V_0)(2)$.

2×2 matrices on this cliff algebra.

$$v + ae_+ + be_- \longmapsto \begin{bmatrix} -v & a-b \\ a+b & v \end{bmatrix}.$$

$$\begin{bmatrix} -v & a-b \\ a+b & v \end{bmatrix}^2 = \begin{bmatrix} v^2+a^2-b^2 & 0 \\ 0 & a^2-b^2+v^2 \end{bmatrix}.$$

$$= \langle v + ae_+ + be_{-1}, \_\_\_ \rangle I.$$

So, $\therefore$ $\Delta V \cong (\Delta V_0)(2)$. $\left(\begin{matrix} 2\times 2 \text{ matrices with} \\ \Delta V_0 \text{ entries.} \end{matrix}\right)$

<u>Fix n</u> $\quad \sigma \mapsto 2-\sigma$ or $\sigma \to \sigma \pm 8$,

$\Rightarrow$ isom Cliff algebras.

$\underbrace{e_1, e_2, \ldots, e_n}$. setup $\rho : V \longrightarrow \Delta V$.

e_1: $e_1^2 = +1$. $\quad e_2, \ldots, e_n$: V_0.

$$ae_1 + v \longmapsto ae_1 + e_1 v.$$

$$(ae_1 + e_1 v)^2 = a^2 - v^2.$$

V_1 with ~~$\langle \; , \; \rangle$~~ $\langle ae_1 + v, ae_1 + v \rangle_1 = a^2 - v^2$.

So, view $\rho : V_1 \to \Delta V$. $\Rightarrow$ $\Delta V_1 \cong \Delta V$.