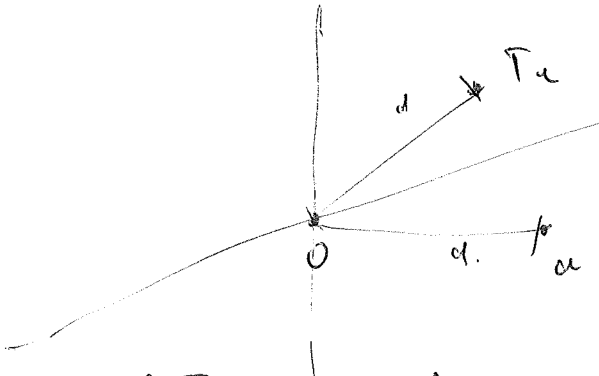
Lecture 5 18/09/2014

Cartan-Dieudonné theorem:

V = Euclidean n-dim, T = isometry
$\exists$ hyperplane reflections $R_1, \dots, R_k$; $k \le n$.
s.t. $T = R_1 \circ \dots \circ R_k$.

Pf. Take $a \in V$: $Ta \neq a$. (if not $T = I$).



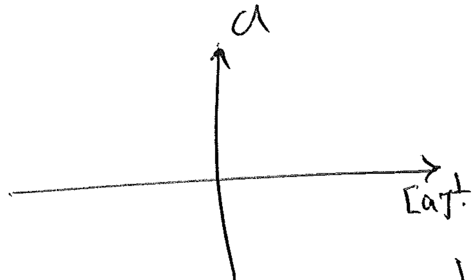
R_1 := hyperplane refl. in $[Ta - a]^\perp$

$|Ta| = |a|$ by isometry property.

$R_1 Ta = a$, fixes $V_0 := [a]^\perp$

Induction on $n \Rightarrow R_1 T = R_2 \cdots R_k$, and $R_1^2 = I$,
so multiply on left.

In terms of a Clifford product ~~Rv~~



$$Rv = a \lrcorner (a \wedge v) - a \wedge (a \lrcorner v)$$

$$\cancel{= -a(a \lrcorner v)}$$

$$= \underbrace{|a|^2}_{1} v - 2\langle a, v \rangle a$$

$$= v - ava - vaa$$

$$= -ava$$

$$= \hat{a} v a^{-1}$$

(Eucl. space distinguished from other inner prod space since ~~$\mathcal{S}$~~ and S do not intersect)

Recall the k-Grassmann Cone $\hat{\Lambda}^k V$. Similarly,

Def^n. (Eucl.).

$$\hat{\Delta} V := \{ a_1 \Delta \cdots \Delta a_n \ ; \forall a_n \in V \}$$ ~~(PIN Group)~~ Clifford Cone.

$\mathrm{Pin} V := \{ q \in \hat{\Delta} V : |q| = 1 \}$.

$\mathrm{Spin} V := \mathrm{Pin} V \cap \Delta^{ev} V$. $\longleftarrow$ Rotors.

Rmk. $O(V)$: {isometries}, $SO(V) =$ {rotations}.

So this is the analogous version in classical terms. $O(V) \to \mathrm{Pin} V$, $SO(V) \to \mathrm{Spin} V$.

"Plücker relations" for $\hat{\Delta} V$ (Plücker relation are f $\hat{\Lambda} V$).

$q \in \Delta V$ belongs to $\hat{\Delta} V \iff q$ invertible and $\forall v \in V$, $\hat{q} v q^{-1} \in V$.

Pf. ($\Rightarrow$) trivial.

($\Leftarrow$) $T : V \to V : v \mapsto \hat{q} v q^{-1}$

T isom!

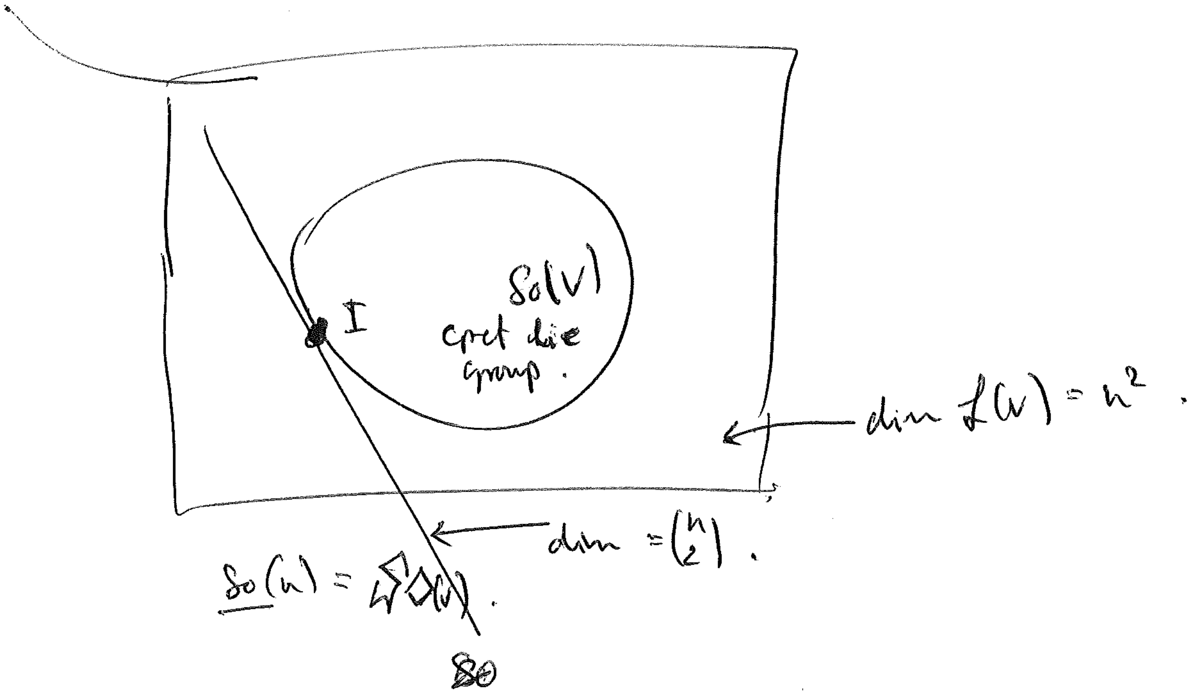
$$|\hat{q} v q^{-1}|^2 = (\hat{q} v q^{-1})(\hat{q} v q^{-1})$$
$$= -\overline{(\hat{q} v q^{-1})}(\hat{q} v q^{-1})$$
$$= -\bar{q}\hat{v}\cancel{\bar{q}^{-1}}\cancel{\hat{q}}v q^{-1}$$
$$= -q \hat{v} v q^{-1}$$
$$= |v|^2 \cancel{q q^{-1}}$$

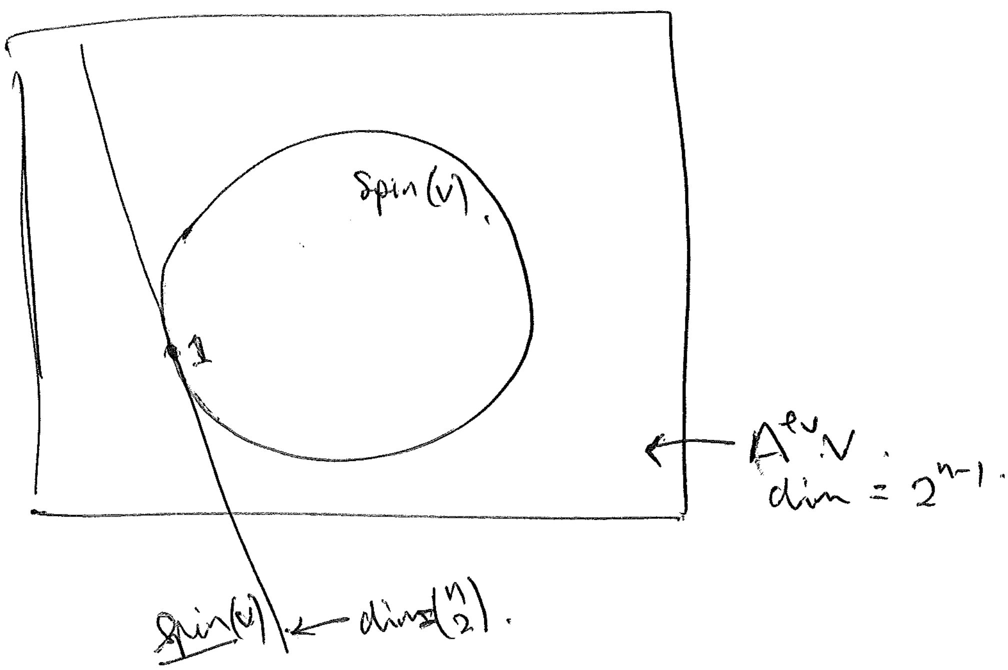
$$q v q^{-1} \overset{cD}{=} \widehat{a_1 \cdots a_n}\, v\, \underbrace{(a_1 \cdots a_n)}_{q_0}^{-1}.$$

$$v(q^{-1}q_0) = \underbrace{\widehat{q^{-1}q_0}}_{\in \Delta V} v \qquad \forall v \in V.$$

$$\Rightarrow \quad q^{-1}q_0 \in \mathbb{R} \quad \Rightarrow \quad q = \lambda q_0 = \lambda a_1 \cdots a_n \in \Delta V.$$

$\mathcal{L}(V)$ — all lin operators on V.





Lie Algebras.

① What is $\underline{so}(V)$? Take $A \in T_I SO(n)$,

$I + \varepsilon A$ is rotation to 1st order

$$|(I+\varepsilon A)v|^2 = |v|^2 + \varepsilon \underbrace{(\langle Av, v\rangle + \langle v, Av\rangle)}_{\overset{\shortparallel}{0}.} + o(\varepsilon^2).$$

$\Longleftrightarrow$ $A^* = -A$. ie skew adjoint.

② What is $\underline{spin}(V)$? $b \in T_1 spin(V)$.

$$\underbrace{(\widehat{1+\varepsilon b})}_{1+\varepsilon b} v \underbrace{(1+\varepsilon b)^{-1}}_{\approx 1-\varepsilon b.} \in ~~V~~spin(V) \text{ to 1st order.}$$

$$v + \varepsilon(bv - vb) + o(\varepsilon^2) \quad \in V.$$

$bv - vb = 2b \llcorner v$. $\quad (\hat{b} = b$ since we're in $\Lambda^{ev} V)$.

$\Rightarrow$ $b \in \Lambda^0 V \oplus \Lambda^2 V$.

If also ~~that~~ $|1+\varepsilon b| = 1$ to first order.

forces $b \in \Lambda^2 V$.

So, $\underline{spin}(V) = \Lambda^2 V$.

Thm.

$$\begin{tikzcd}
b \in \mathfrak{spin}(V) = \Lambda^2 V \arrow[rr, "\text{surj}\ (5)", "q = e^{b/2}"'] \arrow[d, "(1)"', "1\text{-}1"] & & \mathrm{Spin}(V) \ni q \arrow[d, "(2)", "2\text{-}1"'] \\
A \in \mathfrak{so}(V) \arrow[rr, "(3)", "T = e^A \ \text{surj}"'] \arrow[rru, phantom, "(4)\ \text{Diagram Commutes}"] & & SO(V) \quad T: v \mapsto q v q^{-1}
\end{tikzcd}$$

$A: v \mapsto b \llcorner v$.

Pf. (1) $b = e_i e_j$.

$$e_{ij} \llcorner e_k = \begin{cases} 0 & k \notin \{i,j\} \\ e_i & k = j \\ -e_j & k = i \end{cases}$$

$$A = \begin{array}{c c} & \begin{array}{cc} & j \end{array} \\ j \rightarrow & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ & \begin{array}{cc} & \uparrow i \end{array} \end{array}$$

(2) surj: C-D. If $q_1 v q_1^{-1} = q_2 v q_2^{-1}$ $\forall v \in V$.

$$\Rightarrow \underbrace{q_1^{-1} q_2}_{\in \mathbb{R}} v = q_1^{-1} q_2 .$$

So ± 1.

(3) surj?

Spectral thm. $T = \begin{pmatrix} \begin{matrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{matrix} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$

(5)

$$A = \begin{bmatrix} \begin{array}{cc|c} 0 & -\varphi_1 & \\ \varphi_1 & 0 & \\ \hline & & \boxed{} \\ & & & \ddots \end{array} \end{bmatrix}.$$

$\exp(A) = T$.

(4) Given $b \in \Lambda^2 V$, $(*)\ e^{b/2} v e^{-b/2} = e^{A} v$, $\forall v \in V$.

Spectral Thm: $A = \begin{bmatrix} \begin{array}{cc|c} 0 & -\varphi_1 & \\ \varphi_1 & 0 & \\ \hline & & \boxed{} \\ & & & \ddots \end{array} \end{bmatrix}.$

Note: $\forall b \in \Lambda^2 V$, $\exists$ ON-basis s.t.

$$b = \underbrace{\varphi_1 e_{12} + \varphi_2 e_{34}}_{\text{commute!}} + \varphi_3 e_{45} + \dots + \text{—}\quad \lfloor \tfrac{n}{2} \rfloor \text{ terms.}$$

commutativity of these $\{e_{i\,i+1}\}$ basis is

$\Rightarrow$ check $(*)$ for each block is enough!

$$e^{\varphi_1/2\, e_{12}}\, v\, e^{-\varphi_1/2\, e_{12}} = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix}.$$

(5). Surjectivity: Given $q \in \mathrm{Spin}(V)$, obtain a $b \in \underline{\mathrm{spin}}(V)$

from (4) s.t. $q v q^{-1} = e^{b/2} v e^{-b/2}$ $\forall v$.

$\Rightarrow$ $q = \pm e^{b/2}$

$= \pm e^{\frac{\varphi_1}{2} e_{12} + \frac{\varphi_2}{2} e_{34} + \dots}$

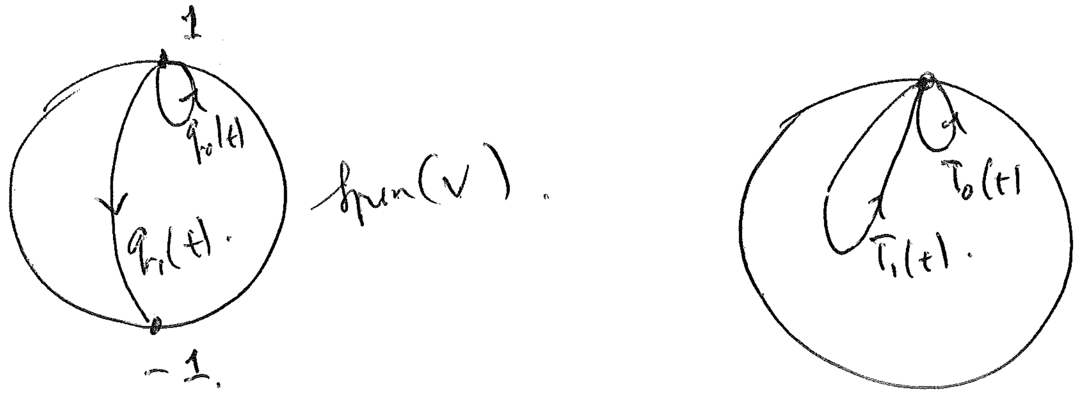
$= (+) e^{(\frac{\varphi_1}{2} + 2\pi) e_{12} + \frac{\varphi_2}{2} e_{34} + \dots}$

Rmk. This works for signature $(-1, 1, \dots 1)$,

But $SO(V)$ is no longer connected. Breaks up via past and future light cones.

- For signatures contain 2 or more $-$ves, fails!.

$Spin(V)$ = universal cover of $SO(V)$, $\dim V \geq 3$, Eucl.



Fix a path $q_1(t) : 1 \mapsto -1$ in $Spin(V)$. ~~Fix~~

~~$q_0(t) : +1 \mapsto +1$~~

$q_0(t) = +1 \quad \forall t \in [0, 1]$.

Let $T_i(t)$ be the corresponding loop in $SO(V)$.

$v \mapsto q_i(t)\, v\, q_i(t)^{-1}$ ~~loop in S~~.

Thm Any loop in $SO(V)$ is homotopic to T_0 or T_1.

Any loop in $Spin(V)$ is homotopic to q_0.

$(\pi_1(SO) = \mathbb{Z}_2 = \{\pm 1\}, \quad \pi_1(Spin) = \{1\})$.

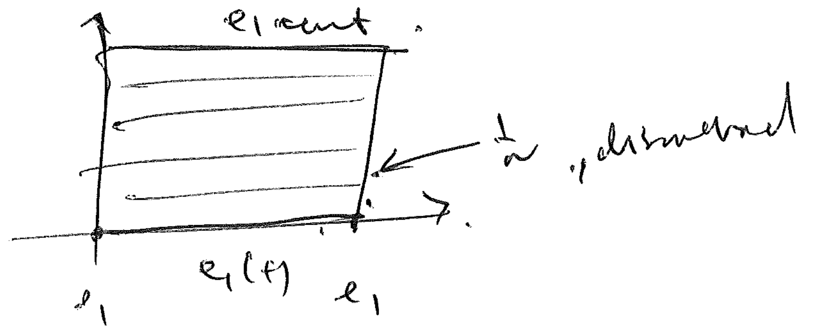
Pf Suffices to show that in $SO(V)$, any loop can be deformed into one of the two given loops.

Given $T(t) = \{e_1(t), \dots, e_n(t)\}$:

Fix $\{e_1, \dots, e_n\}$ o.n. basis.

$e_1(t)$ loop in S^{n-1} – unit sphere in $\mathbb{R}^n$.
(simply connected.

$\Rightarrow$ homotopy $e_1(s,t)$ to e_1.



$e_1(0,t), [e_2(0,t), \dots, e_n(0,t)]$

$\downarrow$ Gram-Schmidt.

$e_1(\frac{1}{N},t), e_2(\frac{1}{N},t), \dots, e_n(\frac{1}{N},t)$

G.S.

$e_1(\frac{2}{N},t)$ $\quad e_2(\frac{2}{N},t), \dots, e_n(\frac{2}{N},t)$.

$T(1,t)$: $e_1(1,t), e_2(1,t), \dots, e_n(1,t)$.

Use $\exp : \mathfrak{so}(V)$ to interpolate $\Rightarrow T(s,t)$ homotopy.
i.e. choose values b/w $\frac{i}{N}, \frac{i+1}{N}$.

$T(1,t)$ ~~path~~ loop in $SO([e_1]^\perp)$ since $e_1(1,t) = e_1$.

Use induction on $\dim V$. Starting point for induction, $n=3$.
In $n=3$, $Spin(\mathbb{R}^3) = S^3 \subset \mathbb{H} \xrightarrow{2:1} SO(V) = \mathbb{RP}^3$.