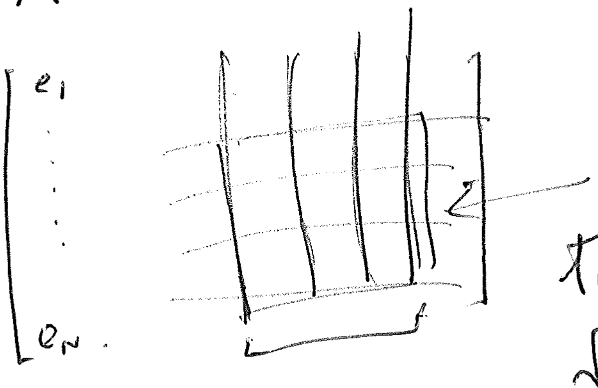
Lecture 4 11/09/2014.

Some motivation for why we look at multilinear.

$f: \mathbb{R}^{3N-4} \longrightarrow \mathbb{R}^{3N}$ volume change?

$$\Lambda^{3N-4}\mathbb{R}^{3N-4} \longrightarrow \Lambda^{3N-4}\mathbb{R}^{3N}$$



delete 4 rows, take det.

This should be same as density

$\sqrt{|(Df)^T Df|}$.

N - really large, corresponds to molecules, problem comes from kinetics.

The Cliff Product:

(X, V) Affine space.

$\langle V, V\rangle$ inner product := symmetric duality.

Lagrange identity: ~~Euclidean Space~~

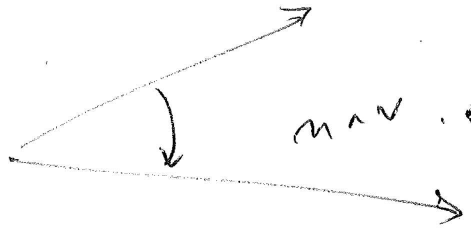
(1) Euclidean Space : $|\langle u,v\rangle|^2 + |u\wedge v|^2 = |u|^2|v|^2$.

Obvious inequality follows from this: Cauchy Schwartz.

Pf: $\langle u\wedge v, u\wedge v\rangle = \langle u, u \lrcorner (u\wedge v)\rangle$.

$u \lrcorner (u\wedge v) = (u \lrcorner u)\, v - (u \lrcorner v)\, u$), check //.

$= \langle u,u\rangle v - \langle u,v\rangle u$.



$u \wedge v \cdot \sin\varphi$.

$\langle u, v \rangle \sim \cos\varphi$.

This identity makes it irresistible to define

$$u \cdot v = u \wedge v + u \lrcorner v. \qquad (u \lrcorner v = \langle u, v \rangle\ ?).$$

<u>Prop 3.4</u>. $\exists!$ Clifford product $\triangle$ on $\wedge V$ s.t.

- $(\wedge V, +, \triangle, 1)$ is an associative algebra.
- $v \triangle w = v \lrcorner w + v \wedge w, \quad \forall v \in V, \ w \in \wedge V$.
- $w \triangle w .= w \llcorner v + w \wedge v$.

<u>Ex</u>. Two vectors $v_1, v_2 \in V$.

$$v_1 \triangle v_2 = \langle v_1, v_2 \rangle + v_1 \wedge v_2.$$
$$v_2 \triangle v_2 = \langle v_1, v_2 \rangle + v_1 \wedge v_2.$$

So, $\frac{1}{2}(v_1 \triangle v_2 + v_2 \triangle v_1) = \langle v_1, v_2 \rangle$. (CAR - canonical anticommutation relation in physics.).

$\frac{1}{2}(v_1 \triangle v_2 - v_2 \triangle v_1) = v_1 \wedge v_2$.

[<u>My note</u>: This looks like solution for $\mathbb{R}$ and $i\mathbb{R}$ parts of a complex number.

when $v_1, v_2 = v$, $v^2 = |v|^2 \in \mathbb{R}$, so,

(I) $\boxed{v^{-1} = \frac{v}{|v|^2}}$

Note: If (1) is non Euclidean, ie., Minkowski, then $|u\wedge v|^2$ may be negative.
Then (I) is valid anyway for <u>null cone</u>.

<u>Ex.</u> Computation in ON-basis, $e_1,\dots,e_n$; $\langle e_i, e_i\rangle = \pm 1$.
depending on signature of inner prod.
Induced basis $\{e_s\}_{s\subset \bar{n}}$ is an ON-basis.

$$\langle e_{s_1,\dots,s_k}, e_{s_1,\dots,s_k}\rangle = \underbrace{\prod \langle e_{s_i}, e_{s_i}\rangle}_{\text{sign}}.$$

(of course $+1$ in Euclidean).

$$e_s \triangle e_t = (e_{s_1}\wedge\cdots\wedge e_{s_k}) \triangle (e_{t_1}\wedge\cdots\wedge e_{t_k}).$$

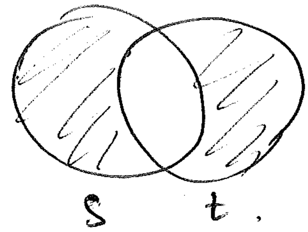
$$= (e_{s_1}\triangle\cdots\triangle e_{s_k}) \triangle (e_{t_1}\triangle\cdots\triangle e_{t_k}).$$

(induction).

move to right position.

$$= \underbrace{\langle e_{s\cap t}, e_{s\cap t}\rangle}_{\pm 1}\ \underbrace{\varepsilon(s,t)}_{\pm 1}.\ e_{s\triangle t}. \quad (*).$$

$$s\triangle t = s\cup t \setminus s\cap t.$$



s t.

~~Term~~ <u>Pf of Prop 3.4</u>

By <u>hypothesis,</u> derive $(*)$. So, we get existence!
ie., assuming uniqueness of Prop. with regards to formulae for $v\wedge w$, ~~and~~

Remains to verify associativity.

This boils down to showing the associativity of symm. set diff $S \Delta t$ and multiplicativity in $\varepsilon(s,t)$. ☒

Formula $v \Delta w$ is not how the ~~full~~ indeed Δ behaves.
I.e., homogen Δ homogen $\rightsquigarrow$ usually inhomogeneous.

- $v \Delta w \in \Lambda^{k-1} V \oplus \Lambda^{k+1} V$, $v \in \Lambda^1, w \in \Lambda^k$.

- $$\underset{\Lambda^2 V}{\underset{\cap}{b_1}} \Delta \underset{\Lambda^2 V}{\underset{\cap}{b_2}} = -\langle b_1, b_2 \rangle + \underbrace{[b_1, b]}_{\text{Cliff commutator?}} + b_1 \wedge b_2.$$

- $\Lambda^k V \Delta \Lambda^l V \subset \Lambda^{|k-l|} V \oplus \Lambda^{|k-l|+2} V \oplus \cdots \oplus \Lambda^{k+l} V$.

(Pf)* ← weak*

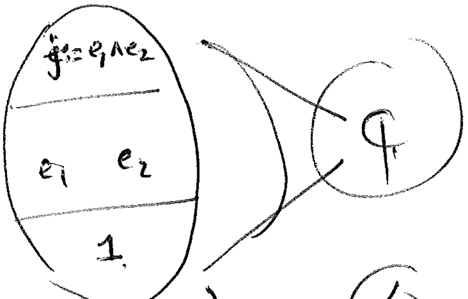
- $e_{12} \Delta e_{12} = e_{12} e_{12} = e_{1212} = -e_{1122} = \underset{\text{Euclidean}}{\underset{\uparrow}{-}} 1 \cdot 1$.
- $e_{12} e_{34} = e_{1234} \in \Lambda^4 V$.
- $e_{12} e_{23} = e_{1223} = e_{13} \in \Lambda^2 V$.

<u>Note:</u> $(\Lambda V, +, \Delta, +1)$ is written as ΔV.

$\Delta V = (\Lambda V, \Delta)$ = hypercomplex number!

V = Euclidean 2-plane:

$q^2 = e_{1212} = -e_{1122} = -1$.



$\Delta^{ev} V = \Lambda^0 V \oplus \Lambda^2 V$ (= $\mathbb{C}$ = std geometric rep).

V = Euclidean 3-space.

Fix orientation, and recall that hodge $*$ commutes in odd dim.

ΛV:

J

$j_1 \; j_2 \; j_3$

$e_1 \; e_2 \; e_3$

1

$\Lambda^{ev} V$.

$$j_1 = * e_1 = e_{23}$$
$$j_2 = * e_2 = e_{31} = -e_{13}.$$
$$j_3 = * e_3 = e_{12} =$$
$$J = e_{123} = * 1.$$

throw away

$$j_1^2 = j_2^2 = j_3^2 = \cancel{(J^2)} = -1.$$

$$j_1 j_2 j_3 = e_{2331\,12} = \underline{+1}$$

Note: In this convention, $\underline{e_i^2 = +1}$

$-1 \to$ maths.

$+1 \to$ physics.

identify $j_1 = -i$, $j_2 = -j$, $j_3 = -k$,

$\Lambda^{ev} V \cong \mathbb{H}$, quaternions.

<u>Two auxiliary operators</u>: on $w = \sum_0^n w_k$, $w_k \in \Lambda^k V$.

$$\bar{w} = w_0 + w_1 - w_2 - w_3 \cdots - w_n.$$
$$= w_0 + w_1 - \sum_{k=2}^n w_k. \quad \text{(Reversion)}.$$

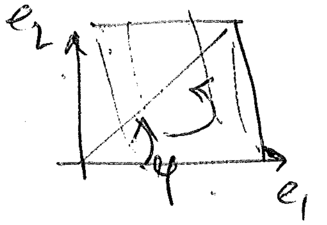
$$\overline{v_1 \wedge \cdots \wedge v_k} = v_k \wedge \cdots \wedge v_1.$$

$$\hat{w} = w_0 - w_1 + w_2 - w_3 + - + \cdots \quad \underline{\text{involution}}.$$

$$\widehat{v_1 \wedge \cdots \wedge v_n} = (-v_1) \wedge \cdots \wedge (-v_n).$$

$v \in [b]$; Fix idea again by considering $b = e_1$, $v = e_{12}$.

$$vb = -vb \Rightarrow e^{-b/2}\, v\, e^{-b/2} = e^{b} v.$$



$$= (\cos\varphi - e_{12}\sin\varphi)\, e_1$$

$$= e_1 \cos\varphi + e_2 \sin\varphi.$$

<u>Uniqueness:</u> $q_1 v q_1^{-1} = q_2 v q_2^{-1}$.

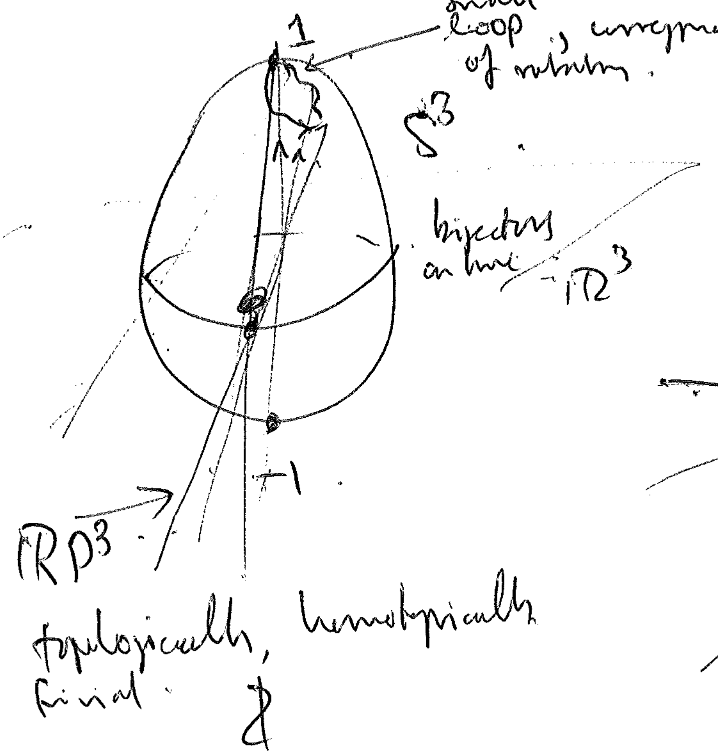
$$(q_2^{-1} q_1) v\, q_1^{-1} q_2 = v (q_2^{-1} q_1) \quad (*)$$

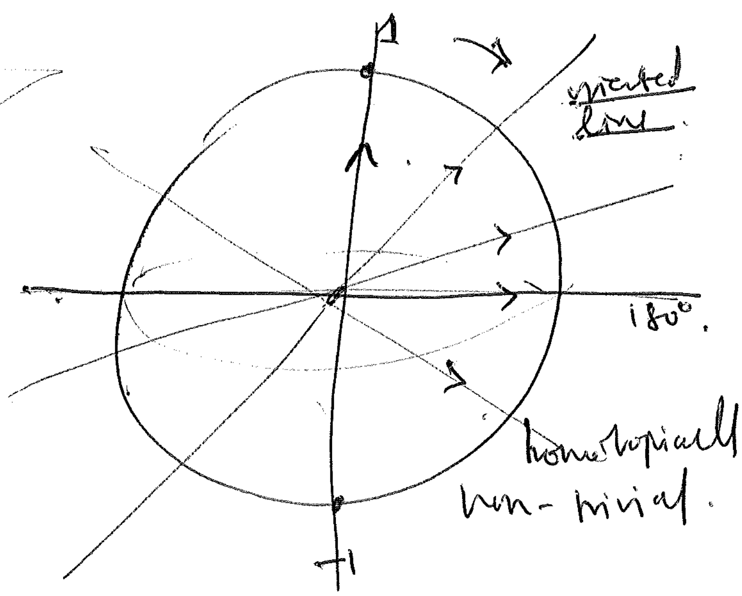
But $q_{q} = q^1 e_{12} + q^2 e_{13} + q^3 e_{23} + q^0$.

But If $q^1, q^2, q^3 \neq 0 \Rightarrow (*)$ violated

So, $(q_2^{-1} q_1) \in \mathbb{R}$.

<u>3D Spin</u>. $\dim_{\mathbb{R}} \mathbb{H} = 4$, so $\mathbb{R}^4 = \mathbb{H}$,





Note: Usually $e_i^2 = -1$ in literature. But here, we allow for any signature so we take $e_i^2 = +1$. The point is that vectors should not be imaginary, but it should be the bivectors. This is why $\bar{w}$ is defined in this way. Andreas says one should not force imaginary in vectors, geometrically speaking; this perspective is correct.

3D Rotations

Let $b \in \Lambda^2 V$. Then, $T : V \to V : v \mapsto e^{b/2} v e^{-b/2}$ is a rotation with rot. plane $[b]$, rotation angle $|b|$ and sense given by b.

If $q_1 = e^{b_1/2}$ and $q_2 = e^{b/2}$ both represent T, then $q_1 = \pm q_2$.

Pf. Fix $b \in \Lambda^2 V$, $q = e^{b/2} = e^{-\varphi/2\, j} = \cos\varphi/2 - j \sin\varphi/2 \in \mathbb{H}$.

$b = -\varphi j$, $|j| = 1$.

$v \perp [b]$: Fix ideas by taking $v = e_3$, $b = e_{12}$.

$$vb = bv \Rightarrow e^{b/2} v e^{-b/2} = e^{b/2} e^{-b/2} v = v.$$