# Lecture 3

08/09/2014.

$V$ linear space, $\dim V = n$, $V^* = \{\theta : V \to \mathbb{R}, \text{ linear}\}$.

$\{e_1, \dots, e_n\}$ basis $\Rightarrow$ dual basis $\{e_1^*, \dots, e_n^*\}$ for $V^*$:

$$e_i^*(e_j) = \langle e_i^*, e_j \rangle = \delta_{ij}.$$

<u>Defn</u>. A map $V_1 \times V_2 \to \mathbb{R}$ $(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$. which is bilinear and non-degenerate is called a <u>duality</u>.

<u>Note</u>: a duality $\langle V', V \rangle$ amounts to a linear invertible map $V' \to V^* : V' \ni v' \mapsto (v \mapsto \langle v', v \rangle)$.

In particular, a duality on $V$, $\langle V, V \rangle$ amounts to an identification $V$ & $V^*$

<u>Examples</u>: (I). Symmetric $\Leftrightarrow$ $\exists$ ON-basis.
(pos Euclidean, neg Lorentzian).

(II) Skew-symm. $\Leftrightarrow$ $\exists$ Darboux

Fix $(X, V)$ affine space and <u>a</u> duality $\langle V^*, V \rangle$.

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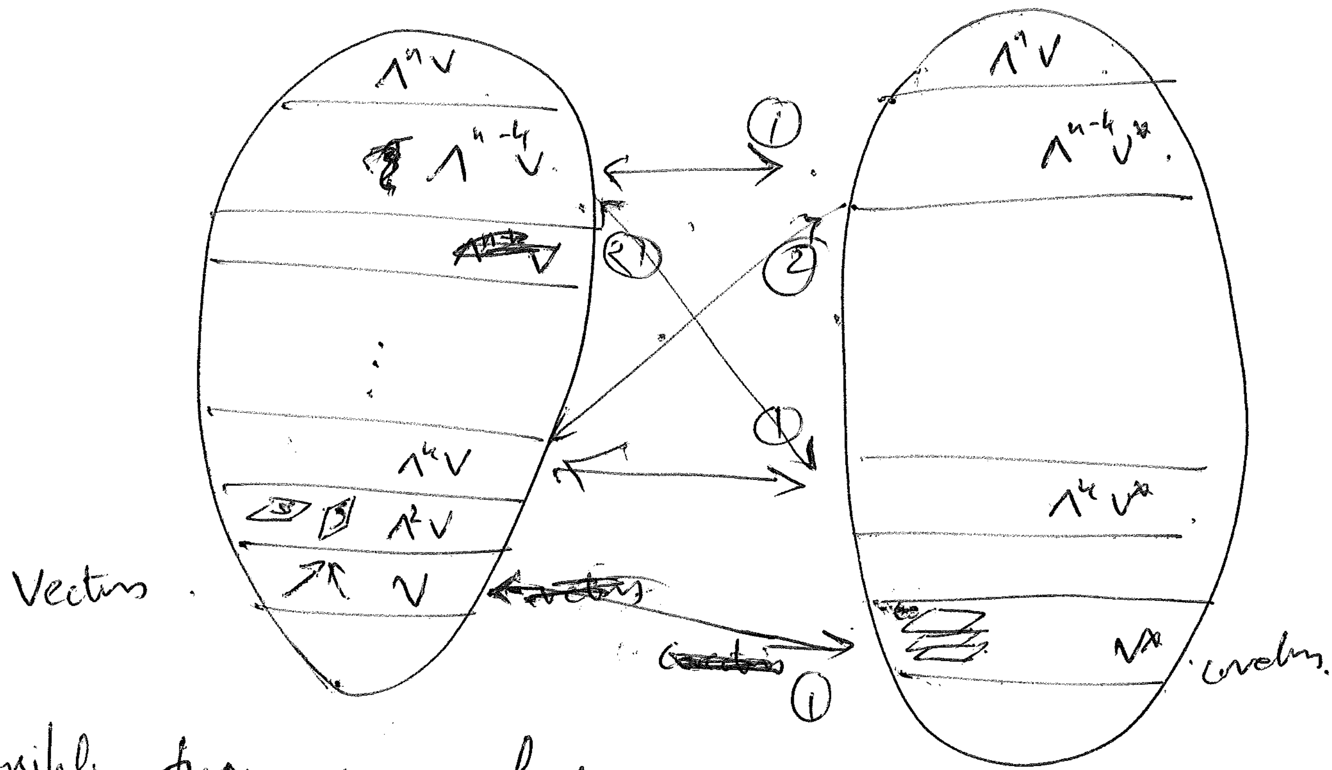
<u>My note</u>: The point here is, later, you want to consider a bundle, and equip a metric. Furthermore, you can compare to the o.n. frame, (coord. independent) to get equiv. of duality.




Two possible types of identification:

② identification through Hodge star duality.
Req. a choice of $0 \neq w \in \Lambda^n V$ ← Requires a volume.

① Identification by duality. ← Requires a metric.

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Induced duality. $\langle \Lambda V^*, \Lambda V \rangle$.

$\{e_i^*\}$ for $V^*$, $\{e_i\}$ for $V$.

$\{e_t^*\}_{t \subset \bar{n}} \longleftrightarrow \{e_s\}_{s \subset \bar{n}}$ basis of $\Lambda V$.

↑ require these to be dual bases.

Note: $\langle \underbrace{\theta_1 \wedge \cdots \theta_k}_{\text{covectors}}, v_1 \wedge \cdots v_k \rangle = \begin{vmatrix} \langle \theta_1, v_1 \rangle & \cdots & \langle \theta_1, v_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \theta_k, v_1 \rangle & \cdots & \langle \theta_k, v_k \rangle \end{vmatrix}$

$\rightsquigarrow \langle \Lambda V^*, \Lambda V \rangle$.

Three points of view on $\Theta \in \Lambda^k V^*$:

(I) k-covector. $\Theta \in \Lambda^k(V^*)$.

(II) linear functional on $\Lambda^k V$, i.e. $\Theta \in (\Lambda^k V)^*$.

(III). Cartan: alternating k-linear map

$$v_1, \dots, v_k \longmapsto \omega(v_1, \dots, v_k).$$

$$V^k \longrightarrow \mathbb{R}.$$

(I) $\Leftrightarrow$ (II) by induced duality.

(II) $\Leftrightarrow$ (III). $\langle \theta, \omega \rangle = \langle \theta, v_1 \wedge \dots \wedge v_k \rangle$ for simple.

But by universal property, it lifts to all $\omega$.

## Interior Product.

My note:

To do $\lrcorner$ ~~with~~ b/w vectors ~~or~~ covectors, need # and b.

This requires a metric!

That's the subtle point!

Left interior product: $\underset{\Lambda^* V}{\theta} \lrcorner \underset{\Lambda V}{\omega} \in \Lambda V$

$$\langle \theta', \theta \lrcorner \omega \rangle := \langle \theta \wedge \theta', \omega \rangle.$$

Right interior product: $\underset{\Lambda V}{\omega} \llcorner \underset{\Lambda V^*}{\theta} \in \Lambda V$

$$\langle \theta', \omega \llcorner \theta \rangle := \langle \theta' \wedge \theta, \omega \rangle.$$

Ex. $\theta = e_j^* \in V^*$

$\omega \in e_S = e_{s_1} \wedge \dots \wedge e_{s_k}$ $\quad S = \{ s_1 < \dots < s_k \}$.

What are $e_i^* \lrcorner e_S$, $e_S \llcorner e_i^*$ ?

$\langle e_t^*, e_i^* \lrcorner e_S \rangle$. $t = \{t_1 < \dots < t_\ell\}$.

$= \sout{\langle e_t^* \wedge e_i^*} \langle e_i^* \wedge e_t^*, e_S \rangle$

$e_i^* \wedge e_t^* = \begin{cases} 0 & , i \in t. \\ \varepsilon(i,t)\, e^*_{\{i\} \cup t} & , i \notin t. \end{cases}$

$\varepsilon(s, t) = (-1)^{\#\{(s_i, t_j) : s_i > t_j\}} = (-1)^{\#\text{ sign changes needed to order increasingly}}$

So, $= \begin{cases} 0 \ \forall t \text{ if } i \notin S. \\ \varepsilon(i, S \setminus \{i\}) \text{ if } i \in S,\ t = S \setminus \{i\}. \\ 0 \quad \text{otherwise.} \end{cases}$

$\therefore \quad e_i^* \lrcorner e_S = \begin{cases} \varepsilon(i, S\setminus\{i\})\, e_{S\setminus\{i\}}, & \text{if } i \in S. \\ 0, & \text{otherwise.} \end{cases}$

$e_i \wedge (e_{s_1} \wedge \dots \wedge e_{s_n})$ add $e_i$ if not present. (with sign).

$e_i^* \lrcorner (e_{s_1} \wedge \dots \hat{e_i} \dots \wedge e_{s_n})$. remove $e_i$ if present. (with sign).

Associative. Contraction. Contraction.

$$(w_1 \wedge w_2) \wedge w_3 = w_1 \wedge (w_2 \wedge w_3).$$

$\Downarrow$

$$(\theta_1 \wedge \theta_2) \lrcorner w = \theta_2 \lrcorner (\theta_1 \lrcorner w).$$

$$w \llcorner (\theta_1 \wedge \theta_2) = (w_1 \llcorner \theta_2) \llcorner \theta_1.$$

$$(\theta_1 \lrcorner w_1) \llcorner \theta_2 = \theta_1 \lrcorner (w \llcorner \theta_2).$$

Commutative

$$w_1 \wedge w_2 = (-1)^{kl} w_2 \wedge w_1. \qquad w_1 \in \Lambda^k V, \; w_2 \in \Lambda^l V.$$

$$\theta \lrcorner w = (-1)^{(k-l)l} w \llcorner \theta. \qquad w \in \Lambda^k V, \; \theta \in \Lambda^l V^*, \; k \geq l.$$

Geometry: If $w_1, w_2$ are simple, $w_1 \wedge w_2 \neq 0$, then $[w_1 \wedge w_2] = [w_1] \oplus [w_2]$.

If $\theta \in \Lambda^k V^*$, $w \in \Lambda^l V$ are simple & $\theta \lrcorner w \neq 0$, then.

$$[\theta \lrcorner w] = [w] \cap [\theta]^\perp =: [w] \ominus [\theta].$$

Andreas' favorite basic formula: Anti commutation relation /

expansion / projections

Th^m 2.75. $\theta \in V^*$, $v \in V$, $w \in \Lambda V$.

$$\theta \lrcorner (v \wedge w) = \langle \theta, v \rangle w - v \wedge (\theta \lrcorner w).$$

Pf. WLOG $\langle \theta, v\rangle = 1$, construct basis $\{e_i\}$

s.t. $v = e_1$, $[\theta]^\perp = \operatorname{span}\{e_2, \dots, e_n\} = N(\theta) =: V_1$

$\Rightarrow$ $\theta^* = e_1^*$. (since $\langle \theta, v\rangle = 1$ and $\langle \theta, e_2\rangle = 0$).

Write. $w = w_1 + e_1 \wedge w_2$, $w_i \in \Lambda V_1$.

$$\theta \lrcorner (v \wedge w) = e_1^* \lrcorner (e_1 \wedge w_1 + \underbrace{e_1 \wedge e_1}_{0} \wedge w_2) = w_1$$

$$v \wedge (\theta \lrcorner w) = e_1 \wedge (\underbrace{e_1^* \lrcorner w_1}_{0} + e_1^* \lrcorner (e_1 \wedge w_2)) = e_1 \wedge w_2$$

$\blacksquare$

~~Expansions of dets.~~

<u>Example</u> (Expansion of dets).

$$\langle e_1^* \wedge \dots \wedge e_n^*, T(e_1 \wedge \dots \wedge e_n)\rangle = \det T.$$

$T(e_1 \wedge \dots \wedge e_n) = (Te_1) \wedge \dots \wedge (Te_n)$.

$\langle e_1^*, Te_1\rangle$ etc, which leads to expansion formula for $\det T$.

<u>Example</u> (projection). $w \in V$, we have:

$$w = \underbrace{\theta \lrcorner (v \wedge w)}_{\text{projection onto } [\theta]^\perp \text{ along } [v]} + \underbrace{v \wedge (\theta \lrcorner w)}_{\langle \theta, w\rangle v}.$$

$\langle \theta, w\rangle v$: projection onto $[v]$ along $[\theta]^\perp$.

## Hodge * duality: Identification type ②.

Fix $0 \neq w \in \Lambda^n V$.

$w = e_{\bar n}$ (= $e_1 \wedge \cdots \wedge e_n$ when we have basis).

dual vectors $e^*_{\bar n} \in \Lambda^n V^*$, $\langle e^*_{\bar n}, e_{\bar n} \rangle = 1$.

$$\theta \longmapsto \theta \lrcorner e_{\bar n} =: \theta *$$

$$\Lambda^{n-k} V \rightleftarrows \Lambda^k V.$$

$$*w := e^*_{\bar n} \llcorner w \longleftarrow w$$

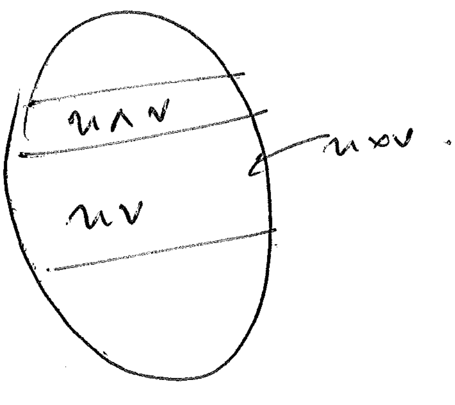
with $\Lambda V$ and $V^*$ interchanged.

- $(*w)* = w$, $*(\theta *) = \theta$.
- $n$ odd $\Rightarrow$ $*w = w*$.
- $w$ simple $\Leftrightarrow$ $*w$ is simple.

  $[*w] = [w]^\perp$.

Example $n = 3$. $V$ is Euclidean, oriented, $e_{\bar 3}$.

$u, v \in V$, $u \times v = *(u \wedge v)$.




$$u \times (v \times w) = *(u \wedge (v \times w))$$
$$= e_{\bar 3} \llcorner (u \wedge (v \times w))$$
$$= (e_{\bar 3} \llcorner (v \times w)) \llcorner u$$
$$= **(v \wedge w)) \llcorner u$$
odd dim.
$$= v \langle w, u \rangle - \langle v, u \rangle w$$

(Th$^m$ 2.75)$'$.

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