Lecture 2   Geometric Multivector Analysis   04/09/2014.

L = linear space, finite dimensional, ~~dim L = n~~.

Distinguish linear space from vector space:

(X, V). Affine space: X affine space, V v.space of affine space, i.e. set of translations $X \to X$.

translations are "vectors".

motivation: k-vectors, they form a linear space, not vector space.

Construction of k-vectors in X, $k = 2, 3, \ldots, n$.

Consider multi-linear maps / k-linear maps:

$$\wedge^k : \underbrace{V \times \cdots \times V}_{V^k} \to L.$$

(A). If $\{v_1, \ldots, v_k\}$ lin. dep., then $\wedge^k(v_1, \ldots, v_k) = 0$.

(B). ~~∄~~ $\exists \{e_1, \ldots, e_n\}$ is a V-basis, ~~then~~ for which

$$\{\wedge^k(e_{s_1}, e_{s_2}, \ldots, e_{s_k})\}_{1 \le s_1 < s_2 < \cdots < s_k \le n}$$

is an L-basis.

~~Only if we ... change diff forms, (B) is automatic. Since $k = \mathbb{R}$. (my note): Note, ... Λ^k ... are the k vectors ... alternating tensors on V.~~

~~• Note~~ $\mathbb{R}$.

Prop 2.3 $\exists \Lambda^k$ sat. (A) and (B).

Pf. Assume (A) & (B). Consider $\{e_1, \ldots, e_n\}$ for (B).

Let $v_j = \sum_i \alpha_{ij} e_i$.

(*) Then,
$$\Lambda^k(v_1, \ldots, v_k) = \sum_{1 \le s_1 < \cdots < s_k \le n} \det \begin{bmatrix} a_{s_1 1} & \cdots & a_{s_1 k} \\ a_{s_k 1} & \cdots & a_{s_k k} \end{bmatrix} \Lambda^k(e_{s_1}, \ldots, e_{s_k}).$$

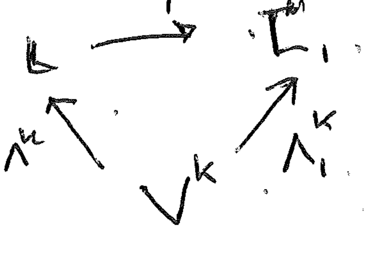
Take L with basis $\{e_S\}_{S \subset \{1, \ldots, n\},\ |S| = k}$.

So, take (*) as def^n with $e_S = \Lambda^k(e_{s_1}, \ldots, e_{s_k})$. where $S = \{s_1 < \cdots < s_k\}$. ∎

(B+) $\forall$ V-bases ... have L-basis.

(U) $\Lambda^k : V^k \to L$ satisfies prop (U) if
$\forall \Lambda_1^k : V^k \to L_1$ mult. lin satisfies (A), then
$\exists!$ lin $T : L \to L_1$ s.t. $\Lambda_1^k = T \circ \Lambda^k$.

Uniqueness is v. important.



(B+) ⇒ (B) ⇒ (U).

(U) ⇒ (B+)

Prop 2.5. If (U) holds for both Λ^k and Λ^k_1, then the T's are inverses.

(U) $\Rightarrow$ (B+): Given $\{e_1, \dots, e_n\}$, construct Λ^k_1 as in Prop 2.3. Then ~~set~~ Λ^k_1 sat. (B), thus (U). So, T is an isomorphism. Since basis $\mapsto$ basis. ∎

Construction of k-vectors

$\Lambda^0 V := \mathbb{R}$.

$\Lambda^1 V := V$.

$\Lambda^2 V := \text{Ran}\,(\Lambda^2 : V^2 \longrightarrow L_2) = L_2$.

$\vdots$

$\Lambda^n V := \text{Ran}\,(\Lambda^n : V^n \longrightarrow L_n) = L_n$.

This is not the range, but rather the span of the range of basis elements.

and $\Lambda V := \bigoplus_{k=0}^{n} \Lambda^k V = \Lambda^0 V \oplus \Lambda^1 V \oplus \dots \oplus \Lambda^n V$.

Algebra on ΛV: $(\Lambda V, +, \wedge)$.

Want: $\wedge$ to satisfy $v_1 \wedge \dots \wedge v_k = \Lambda^k(v_1, \dots, v_k)$.
and extend to all of ΛV.

Properties:
- bilinear
- associative
- anti-comm. $w_1 \wedge w_2 = (-1)^{k \cdot l}\, w_2 \wedge w_1$.

Ex: Rectangular determinant:

$(a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_2)$.

So, how do we compute? $\rightsquigarrow$ $\det \begin{bmatrix} e_1 & a_1 & b_1 \\ e_2 & a_2 & b_2 \\ e_3 & a_3 & b_3 \end{bmatrix}$.

$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} e_1 \wedge e_2 + \cdots$.

If $k = n$:

$\left| \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix} \; \square \right| = \left| \square \right| \underbrace{e_1 \wedge \cdots \wedge e_n}_{e_{\bar n}}$.

In general? $\to$? sub-determinants?

Linear maps.

Given linear $T : V_1 \to V_2$, consider the k-linear map.

$v_1, \ldots, v_k \mapsto (Tv_1) \wedge \cdots \wedge (Tv_k) \in \Lambda^k V_2$.

(U) for $\Lambda^k V_1 \Rightarrow \exists!\; T_k : \Lambda^k V_1 \to \Lambda^k V_2$ s.t.

$$T_k(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k).$$

$T_\wedge : \Lambda V_1 \to \Lambda V_2$,

$$T_\wedge := I \oplus T \oplus T_2 \oplus \cdots \oplus T_n.$$

map of $\wedge$ induced by T.

Note: $T_n(w) = \det T \cdot w$ if $V_1 = V_2$.

T_k are given by sub-determinants.

$(TT')_\wedge = (T_\wedge)\cdot(T'_\wedge)$.

$(\alpha T)_k = \alpha^k T_k$.

$(T+T')_\wedge = \; ? \; \neq T_\wedge + T'_\wedge$.

↑ in general

Simple / composite k-vectors.

$$\wedge V \supset \wedge^k V \supset \hat{\wedge}^k V =: Ran(\wedge^k).$$

(under $\wedge^k V$:) homogeneous k-vectors

(under $\wedge V$:) ↑ inhomogeneous multivectors.

$\hat{\wedge}^k V$ is not a linear space since $\wedge^k$ is multi-linear. Rather this is a cone. This is the Grassmann cone.

$\hat{\wedge}^k V$ – simple

$\wedge^k V \setminus \hat{\wedge}^k V$ – composite.

inhomogeneous $\neq$ composite.

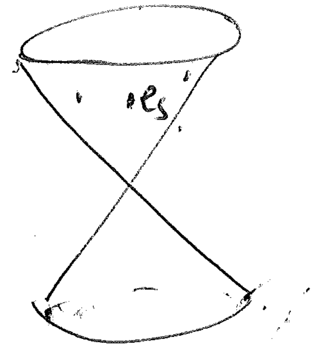
My Note: $\wedge^k$ maps $Sc\{e_{s_1}, \dots e_{s_k}\}_{1 \le s_1 < \dots < s_k \le n}$ ~~[illegible]~~ to basis elements in $\wedge^k V$, however the $Ran(\wedge^k) \neq \wedge^k V$.

Cone: $\lambda(v_1 \wedge \dots \wedge v_k)$.

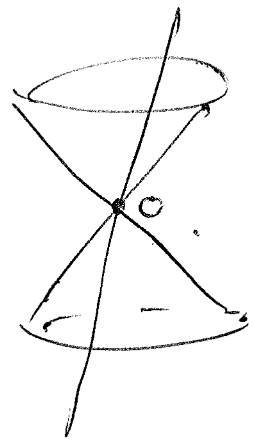
$= (\lambda v_1) \wedge \dots v_k$.

$= v_1 \wedge \dots \wedge (\lambda v_k)$

all basis elements in ~~cone~~ Grassmann cone.



V		$\Lambda^k V$.
$\{v_1, \dots, v_k\}$ lin indep.	$\Leftrightarrow$	$v_1 \wedge \dots \wedge v_k \neq 0$.
$V' \subset V$ k-dim.	$\Leftrightarrow$	a line in $\hat{\Lambda}^k V$ through 0.



Problem: Given a k-vector.

$$w = \sum_{\substack{S \subset \bar{n} \\ |S| = k}} \alpha_S e_S .$$

$$e_S = e_{\{s_1 < \dots < s_k\}} = e_{s_1} \wedge \dots \wedge e_{s_k}.$$

how do we determine if w is simple and if so, how do we factorise it?

Def: $w \in \Lambda^k V$

$\llcorner w \lrcorner := \{ v \in V : v \wedge w = 0 \}$ inner space.

$\left(\ulcorner w \urcorner := \bigcap \{ V' \subset V : w \in \Lambda^k V' \},\ V' \text{ subspace} \} \right)$

~~subspace~~ outer space.

$\omega = v_1 \wedge \dots \wedge v_k$ (simple).

- ~~Then~~ $\lfloor \omega \rfloor = \lceil \omega \rceil = \mathrm{span}\{v_1, \dots, v_k\} =: [\omega]$.

  Def^n.

- If ω_1, ω_2 simple and $\omega_1 \wedge \omega_2 \neq 0$, then $\omega_1 \wedge \omega_2$ simple and

  $$[\omega_1 \wedge \omega_2] = [\omega_1] \oplus [\omega_2].$$

- $\lfloor 0 \rfloor = V$, $\lceil 0 \rceil = \{0\}$ (since 0 is contained in every subspace).

- In general, for $\omega \neq 0$: $\lfloor \omega \rfloor \subset \lceil \omega \rceil$.
  (the exception is $\omega = 0$ as above!)

(1) $\dim \lfloor \omega \rfloor \leq k$ with $=$ iff ω is simple.

(2) $\dim \lceil \omega \rceil \geq k$ with $=$ iff ω is simple.

($\Lambda^2 \mathbb{R}^4$, $e_1 \wedge e_2 + e_3 \wedge e_4$ is <u>not</u> simple!).

(<u>My note</u>: $\omega = v_1 \wedge \dots \wedge v_k \neq 0$ is a k-plane, say $V = T^pM$ (or TM). Recall Brothers saying how to associate ~~k-planes with k-vec~~ oriented k-planes with k-vectors.).

Factorisation of k-vectors

$0 \neq w \in \Lambda^k V$.

$\lfloor w \rfloor = \text{span}\{v_1, \dots, v_k\} \subset \{v_1, \dots, v_n\}$ V-basis.

$V' := \text{span}\{v_2, \dots, v_n\}$.

Lemma 2.27

$$\Lambda^{k-1} V' \longrightarrow \{u \in \Lambda^k V : v_1 \wedge u = 0\}.$$

$$\tilde{w} \longmapsto v_1 \wedge \tilde{w}.$$

is an invertible linear map.

Pf. $\underbrace{v_1 \wedge v_1 \wedge \tilde{w}}_{\substack{\shortparallel \\ 0}} = 0$. $\Rightarrow$ bijective.

Injective: $w = w_1 + v_1 \wedge w_2$, with $w_1 \in \Lambda^k V'$, $w_2 \in \Lambda^{k-1} V'$.

$0 = v_1 \wedge w = v_1 \wedge w_1 \Rightarrow w_1 = 0$. ◻

So, we can write $w = v_1 \wedge \tilde{w}$, ~~hence~~ $\tilde{w} \neq 0$ since $v_1 \wedge w = 0 \Rightarrow w \in \{u \in \Lambda^k V : v_1 \wedge u = 0\}$.

$\lfloor \tilde{w} \rfloor \subset \lfloor w \rfloor$: $v \wedge \tilde{w} = 0 \Rightarrow v \wedge w = v_1 \wedge v \wedge \tilde{w} = 0$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v_1 \wedge \tilde{w}$.

$v_1 \notin \lfloor \tilde{\omega} \rfloor$, $v_2, \dots, v_\ell \in \lfloor \tilde{\omega} \rfloor$

$v_j \wedge \tilde{\omega} \overset{?}{=} 0$

$0 = v_j \wedge \omega = -v_1 \wedge \underbrace{(v_j \wedge \tilde{\omega})}_{\in \Lambda^k V'}$. $v_j \wedge \tilde{\omega} = 0$ by lemma.

$\Rightarrow \lfloor \tilde{\omega} \rfloor = \operatorname{span}\{v_2, \dots, v_\ell\}$.

Then, by iteration, $\omega = (v_1 \wedge v_2 \wedge \dots \wedge v_\ell) \wedge \omega_0$.

So, since $0 \neq \omega \in \Lambda^k V$, we must have

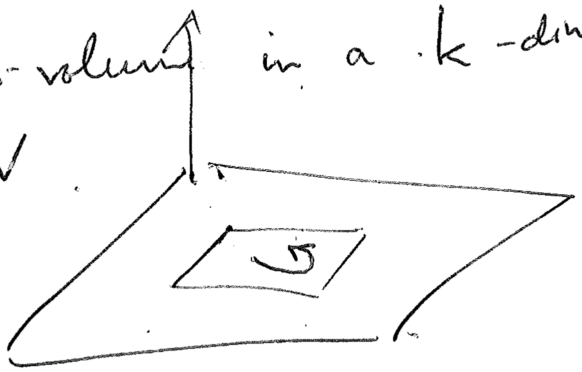
$\dim \lfloor \omega \rfloor \leq k$. ∎

<u>Geometric interpretation of multivectors:</u>

inhom: ?

(1). simple k-vector:

$\omega \sim$ a k-volume in a k-dim subspace, $V' \subset V$



(2) composite k-vector: $f: D \subset \mathbb{R}^k \to X$.

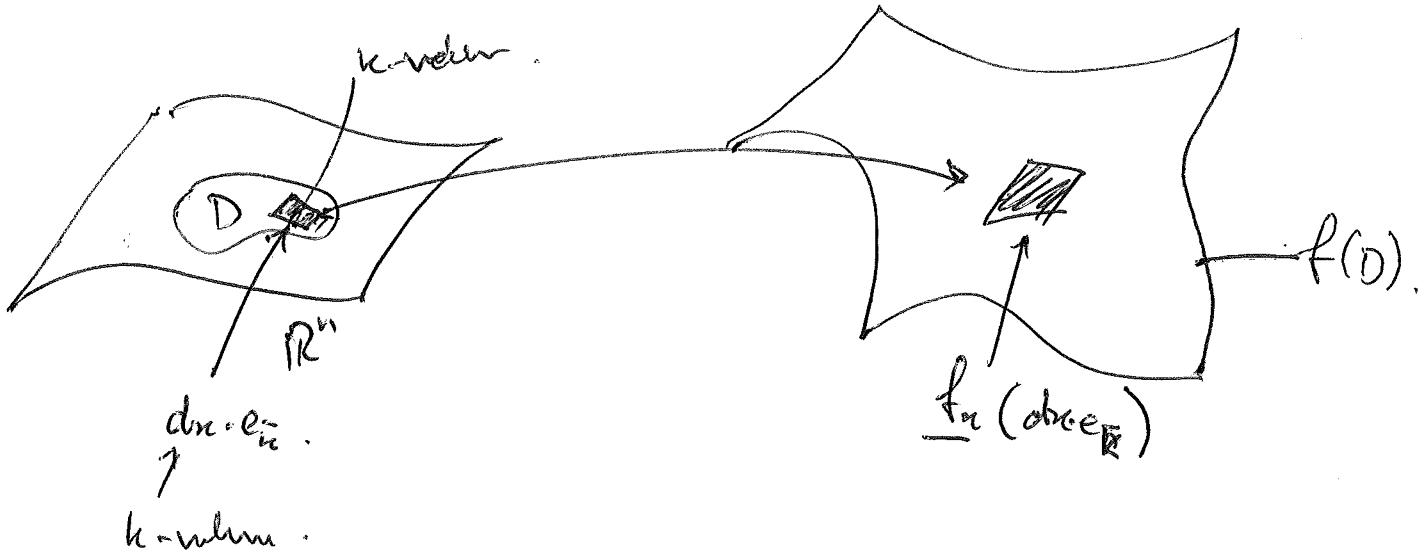
$f(D)$ k-surface in X.

diff / Jacobian / tot. derivative.

$f_x : \mathbb{R}^k \to V$ $\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$

induced linear map

$$\underline{f_x} : \Lambda^j \mathbb{R}^k \to \Lambda^j V.$$



Def^n. oriented measure of $f(D)$.

$$= \int_D \underline{f_x}(dx\, e_{\bar{k}}) = \int_D \underline{f_x}(e_1) \wedge \dots \wedge \underline{f_x}(e_k)\, dx.$$

Note: $\{\underline{f_x}(e_k).\}$ need not be a basis.