Tuomas Hytönen 03/10/2014.

2WP for HT.

I/ History.

1905 : D. Hilbert. $H(f\,d\sigma)(x) := \int \frac{f(y)}{x-y}\,d\sigma(y).$
$x \notin \operatorname{spt} f.$

1920's . M. Riesz

$$\|H(f\,d\sigma)\|_{L^p(\sigma)} \leq c\,\|f\|_{L^p(\sigma)}. \quad 1<p<\infty$$
$$\sigma \in \{ \mathcal{H}^1|_{\mathbb{T}},\ \mathcal{H}^1|_{\mathbb{R}},\ \textstyle\sum_{k\in\mathbb{Z}} \delta_k \}.$$

3WP $\|H(f\,d\mu)\|_{L^p(\nu)} \leq c\,\|f\|_{L^p(\lambda)}.$

wlog. $\underline{\mu \ll \lambda \ll \mu}.$

$$\Longleftrightarrow \quad \|H(f\underbrace{\varphi\,d\mu}_{d\sigma})\|_{L^p(\nu)} \leq C\|f\varphi\|_{L^p(\lambda)} = C\|f\|_{L^p(\underbrace{\varphi^p d\lambda}_{d\sigma})}.$$

$$\varphi\,d\mu = \varphi^p\,d\lambda. \quad \varphi^{p-1} = \frac{d\lambda}{d\mu}.$$

So, 3WP $\Leftrightarrow$ 2WP.

2WP: $\|H(f\,d\sigma)\|_{L^p(\nu)} \leq C\|f\|_{L^p(\sigma)}$

$$\Longleftrightarrow \quad \|H(g\,d\nu)\|_{L^{p'}(\sigma)} \leq C\|g\|_{L^{p'}(\nu)}.$$

1970's: 1WP

$$\| H(f\,dx)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

$\Leftrightarrow$ $\| H(f\,d\sigma)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}$ $\sigma = w^{-\frac{1}{p-1}}$ $= \frac{1}{w}$ $p=2$.

$\Leftrightarrow$ $\sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I \sigma\right)^{p-1} =: [w]_{A_p} < \infty$

$\Leftrightarrow$ $(\star)$ for Maximal M in place of H.

$\Leftrightarrow$ " ———— T any $\epsilon > 0$.

A_p condition really a characterisation.

Classical 2WP:

$$\| H(f\,dx)\|_{L^p(\sigma)} \lesssim C\|f\|_{L^p(\lambda)}.$$

But recall that. $1 \lessgtr\!< dx << 1$.

But in modern 2WP, bit more general and can allow for singular measures.

$p=2$ well understood, but anything else is quite open.

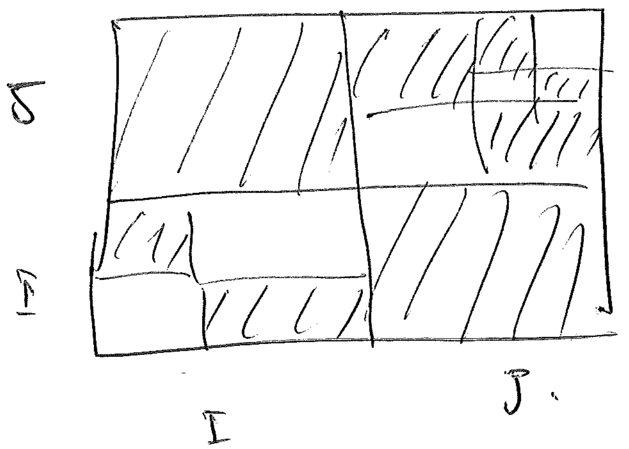
$$\| H(f\,d\sigma)\|_{L^2(w)} \leq C\|f\|_{L^2(\sigma)}$$ ($w, \sigma << dx$, so, really weights).

$$\Rightarrow [w,\sigma]_{A_2} := \sup_I \frac{\sigma(I)}{|I|}\frac{w(I)}{|I|} < \infty$$

Equiv

$$\frac{\sigma(I)\, w(J)}{|I|\,|J|} \le C. \quad I, J \text{ adj, equal length.}$$

$$\sigma \times w(I \times J) \le C\, |I \times J| \;\leftarrow\; \text{way to interpret}$$



$[w,\sigma]_{A_2} \not\Rightarrow$ Hilbert hold.

$$[w,\sigma]^{**}_{A_2} = \sup P(\sigma, I) \cdot P(w, I) < \infty.$$

$$P(\sigma, I) = \int_{\mathbb{R}} \frac{|I|}{|I| + (x - c_I)^2}\, d\sigma(x).$$

$\not\Leftarrow$. Sarason's Conjecture, disproven by F. Nazarov.

But if we take

$$\sup_I \left(\frac{1}{|I|} \int_I \sigma^r\, dx \right)^{\frac{1}{r}} \left(\frac{1}{|I|} \int_I w^r\, dx \right)^{\frac{1}{r}} \Rightarrow \text{sufficient.}$$

$r > 1$. Neugebauer.

Testing paradigm

Want inequality for all functions, but enough to consider it for a subcollection.

(*) for all $f \in \mathcal{F} \overset{?}{\Rightarrow}$ ~~(*)~~ for all $f \in L^2(\sigma)$.

Roots: 1950 E Sawyer solved 2WP for several positive T

David-Journé T1 $T \in CZO$, then

$$T: L^2 \to L^2 \iff \|T 1_I\|_2 \le C|I|^{\frac{1}{2}}.$$
$$\|T^* 1_I\|_2 \le C|I|^{\frac{1}{2}}.$$

$\mathcal{F} = \{1_I : I \text{ intervals}\}$.

connected to

Nazarov-Treil-Volberg ~ 2000 ⇒ Noticed 2WP.

Lacey-Sawyer-Uriarte-Tuero: ~2010 ⇒ Pivotal condition!

Thm (L-S-Shen-U-T)

σ, ω Radon. w/o common atoms: $\sigma(\{a\})\,\omega(\{a\}) = 0$.

Then TFAE $\iff$

$$\begin{cases} \|H(1_I\sigma)\|_{L^2(\omega)} \le C\,\sigma(I)^{\frac{1}{2}}. & (a) \\ \|H(1_I\omega)\|_{L^2(\sigma)} \le C\,\omega(I)^{\frac{1}{2}}. & (b) \\ \sup_I P(\sigma, I)\,P(\omega, I) < \infty & \end{cases}$$

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Actually. $\iff$ $\begin{cases} \text{w/o common atoms} \\ \text{bdd Hilbert Transform.} \end{cases}$ 2wp.

(another way of stating the thm).

<u>T. H. version</u>:

σ, w Radon.

H-T. bdd. $\iff$ $\begin{cases} \text{(u)} \\ \text{(b)} \end{cases}$ (from prev thm).

$\cancel{\sup_I P(\sigma, I) P(w, I) < \infty}$ ← this is what causes the

$$[w,\sigma]^*_{A_2} = \sup_I \frac{w(I)}{|I|} \int \frac{|I|}{|I|^2 + (x - c_I)^2} \, d\sigma(x) < \infty.$$

$$[\sigma, w]^*_{A_2} < \infty.$$