Tao Mei 02/04/2014.

$\|\nabla_L f\|_p \simeq \|Lf\|_p$, L given, what is ∇_L.

P.A. Meyer's Gradient form.

$$2\Gamma(f_1, f_2) = -L(\bar{f}_1 f_2) + L(\bar{f}_1) f_2 + \bar{f}_1 L(f_2).$$

Ex. ① $L = \Delta$, Laplace Beltrami.

$\Gamma(f,f) = |\nabla f|^2$.

② (M, dm) complete Riem mfld

$Lf(x) = \sum_i a_{ij}(x) \partial_i \partial_j f + \sum_i g_i(x) \partial_i f$.

$\Gamma(f,f) = \sum_i a_{ij} \partial_i \bar{f} \partial_j f$.

③ $L = \Delta^{\frac{1}{2}}$, $S_t = e^{-tL}$ on $\mathbb{R}^n$.

$\Gamma(f,f) = \int_0^\infty |\nabla S_t f|^2 + |\partial_t S_t f|^2 \, dt$.

Γ Carré du Champ?

Q. Can we use this for higher energies?

Eric Ricard

02/10/2014.

$M \subset B(\ell_2)$ von Neumann Algebra ; faithful trace.

$$(M,\tau) \longleftrightarrow (L_\infty(\Omega),\mu).$$

$$\tau(xy)=\tau(yx) \longleftrightarrow fg=gf$$

normal $\longleftrightarrow$ dominated convergence.

faithful $\longleftrightarrow$ full support of integrals.

$$\underbrace{L_0(M,\tau)}_{\text{unbounded ops.}} \longleftrightarrow \underbrace{L_0(\Omega,\mu)}_{\text{unbounded functions.}}$$

$$L_p(M,\tau)=\{ f\in L_0(M,\tau) : \|f\|_p^p < |\tau(x)|^{(p?)} \}.$$

<u>Difficulties</u>: $\nexists C>0$

$$\| |x| - |y| \|_1 \leq C \|x-y\|_1 .$$

I.e., abs value is <u>not</u> Lipschitz in non-com.

Stephen Fackler. 02/10/2014.

— A generator of bdd analytic C_0-semigroup on X B-space.

Maximal Reg.: $0 \neq s \mapsto is R(is, A)$ defines

bdd Fourier multiplier on $L_p(\mathbb{R}; X)$.

for one (equiv all) $p \in (1, \infty)$.

Lutz Weis: — A ~~positive~~ on L^p, positive and contractive, on $\mathbb{R} \Rightarrow A$ has maximal regularity.

Q.. Is this abstract? Ie, $L^p(\mathbb{R}^n)$ or more general L^p?

$p=2$. always maximal regularity

Can Reisz transform question be phrased via this?

What about stability of Hodge projections?

Markus Haase. 02/10/2014.

I/ What is a FC?

Heuristic: $\Phi : \mathcal{F} \longrightarrow \{ \text{op's on } X \}$.

$\updownarrow$ "computation script".

Some algorithm we want to apply to operators.

Formal: $\mathcal{F}$ function algebra.

Φ Alg. homom.

~~One two~~.

$$\left(\int_\Omega f(t,z)\, d\mu(t) \right)(A) \overset{?}{=} \int_\Omega f(t,A)\, d\mu(t).$$

II. How does one capture fundity?

$\rightsquigarrow$ Elementary Calculus.

A sectorial: (not nec. inj).

$$\Phi(f) = f(A) = \frac{1}{2\pi i} \oint_{\partial S_\omega} f(z) R(z,A)\, dz.$$

Which f? $H_0^\infty(S_\mu)$. $\leftarrow$ but too much smoothness? (2 class).

automatic

Instead: $\mathcal{E}_0(S_\mu) = \left\{ f \in H^\infty : \int_{\partial S_\alpha} |f(z)| \frac{|dz|}{|z|} < \infty \quad \forall 0<\alpha<\varphi \quad \cancel{\lim_{z\to 0} f(z) = 0} \right\}$

$\mathcal{E}_0(S_\mu)$ only has integrability condition.

$$\mathcal{E}(S_\mu) = \mathcal{E}_0(S_\mu) \oplus \frac{\mathcal{F}}{Hz} \oplus \mathcal{F}\mathbb{1}. \quad \text{(algebraic sum)}.$$

"Elementary Class".

III. Alg. Ext. $\to$ motivated by alg. homom. prop.

$\Phi : \mathcal{E} \to \mathcal{L}(x)$.

$\mathcal{E} \subset \mathcal{F}$, $f \in \mathcal{F}$, $[f] = \{e \in \mathcal{E} : ef \in \mathcal{E}\}$.

Def^n. f regul. if $\bigcap_{e \in [f]} \ker e(A) = \{0\}$.

$$f(A)x = y \overset{\text{def}}{\iff} (ef)(A)x = e(A)y.$$

If $\exists e$, $e(A)$ inj, $f(A) = e(A)^{-1}(ef)(A)$.
"f strictly reg"

Open: φ H-P calc. for hdd semigroups, if f reg.
$\Rightarrow$ f strictly reg?

[True for bdd groups]

Note: $\int_\Omega f(t,A)\, d\mu(t)$ might be well defined, but
$\int_\Omega f(t,z)\, d\mu(t)$ may not be in the F.C.

leads to (IV).

(IV). Top. Ext.

$\Phi: \mathcal{E} \to \mathcal{L}(\mathcal{X}). \quad \mathcal{E} \subset \mathcal{F}.$

- conv. notion in $\mathcal{F}$.
- " " " $\mathcal{L}(\mathcal{X})$.

] algebraic notions are ctt. ~~wrt~~ w.r.t. the top.

- $e_n \to 0$, $e_n(A) \to T \Rightarrow T = 0$.

$$\Rightarrow \mathcal{E}_{top} = \{ f : \exists (e_n) \subset \mathcal{E}, \ e_n \to f, \ \exists T. \ e_n(A) \to T \}$$

define $f(A) = T$.

Example: A sect, $f \in \mathcal{E}(S_\varphi)$

$\mu \in Meas(\mathbb{R}_+)$.

$$g(z) = \int_0^\infty f(tz) \, d\mu(t).$$

$\Rightarrow g \in \mathcal{E}_{top}(S_\varphi)$.

$$g(A) = \int_0^\infty f(tA) \, d\mu(t).$$

$$f = \frac{1}{1+z} \Rightarrow \int_0^\infty (1+tA)^{-1} \, d\mu(t).$$

Further: $\dfrac{1}{1-\log z} = \int_0^\infty \dfrac{1}{(1+\log t)^2+\pi^2}\, \dfrac{1}{(1+tz)}\, \dfrac{dt}{t}$.

is not regularisable, but accessible via Ext.

(V). What is a SF (E)?

$X = L^p$, $\quad H = L_2^*\left((0,\infty), \frac{dt}{t}\right)$.

$$\|x\|_X \gtrsim \Big\| \underbrace{\left(\int_0^\infty |f(tA)x|^2\right)^{\frac{1}{2}}}_{} \Big\|_X .$$

K-W.

$$\simeq \Big\| \left(\sum_{n=1}^\infty |[T_f x]\, a_n|^2\right)^{\frac{1}{2}} \Big\| .$$

(a_n) onb of H.

$T_f x : H \to X$, $\quad h \mapsto \int_0^\infty h(t)\, f(tA)x \, \frac{dt}{t}$.

$$\simeq \left(\mathbb{E} \Big\| \sum_{n=1}^\infty \gamma_n \otimes [T_f x]\, a_n \Big\|_X^2\right)^{\frac{1}{2}} .$$

$$= \|T_f x\|_{\gamma(H;X)}$$

γ-radonifying ops.

Def^{n}. SF

$T : \mathrm{dom}(T) \subset X \to \gamma(H; X)$.

x.

4

$$[T_f x]h = \int_0^\infty h(t) f(tA)x \frac{dt}{t}.$$

$$= \int_\Omega h(t) f(t,A)x \, d\mu(t).$$

$$\boxed{\begin{aligned} f = f(t,z) &: S_\varphi \to H' \\ f &\in H^\infty(S_\varphi, H'). \end{aligned}}$$

dual of H-space, better keep track of dual.

$$= \left(\int_\Omega h(t) \, f(t,z) \, d\mu(t)\right)(A) x.$$

spirit of F.C.

$$= \Phi\left(\langle f(z), h\rangle_{H' \times H}\right) x.$$

$$=: \underbrace{[\Phi_\gamma(f) x]}_{\in\, \gamma(H;X)} h.$$

$$\operatorname{dom} \Phi_\gamma(f) = \{x \in X : \forall h \in H, \ x \in \operatorname{dom} \Phi(\langle f(z), h\rangle)$$
$$\text{and } h \mapsto [\Phi_\gamma(f)x]h \in \gamma(H;X)\}.$$

$$\Phi_\gamma : H^\infty(S_\mu; H) \longrightarrow \text{SF's}.$$

"vectorial" F.C.

(IV) Subordination

$T : X \to \gamma(H; Y)$.

$M : K \to H$, K, H Hilbert spaces.

$T_M : X \to \gamma(K; Y)$.

$T_M x = T_x \circ M$.

$T_M \lesssim T$ "subordinate".

$S \simeq T$ if $S \lesssim T$, $T \lesssim S$.

Example. • A strip-type.

• shift type. $\varphi(t+z)$; $H = L^2(\mathbb{R})$.

• $$\frac{\pi/\omega}{\cosh\left(\frac{\pi}{2\omega}(t+z)\right)} \overset{F.T.}{\simeq} \frac{e^{-isz}}{\cosh(\omega s)}$$

$$\simeq e^{-\omega|s|} e^{-isz} \simeq \left(\mathbb{1}_{\mathbb{R}_+} e^{-\omega s} e^{+isz}\right).$$

$$\simeq \frac{1}{\pm i\omega + t - z}.$$

Th^{m}. $T : X \to \gamma(H; Y)$, Y property (α).

$\{T_M : \|M\| \le 1, M \in \mathcal{L}(K; H)\} \subset \mathcal{L}(X, \gamma(K; Y))$.

$\Rightarrow$ γ-bounded.

Bernard Haak. 02/10/2014.

Φ_γ

$$\Phi_\gamma : \begin{cases} X \longrightarrow \gamma(H;X). \\ x \longmapsto h \longmapsto h(F(z))_H(A)x. \end{cases}$$

Hyps
- X cotype space.
- A had $H^\infty(\vartheta)$.

$$\mathbb{E}\,\|\sum \gamma_n \langle h_n | F(t)\rangle (A)x\|.$$

$$\sim \mathbb{E}\,\|\sum r_n \langle h_n | F(z)\rangle (A)x\|.$$

$$= \mathbb{E}\,\|(\sum r_n \langle h_n | F(z)\rangle)(A)x\|.$$

$$\lesssim C_H \|x\| \sup_z \sum_n |\langle h_n | F(z)\rangle|.$$

<u>Trick</u> $R : H \to \ell_2$, $L : \ell_2 \to H$.

$T = \Phi_\gamma(F)x.$ $LR = I$

$$\|T\|_{\gamma(H;X)} = \|T \circ R^* L^*\|_{\gamma(H;X)} \leq \|L\|\,\|T \circ R^*\|_{\gamma(\ell_2;X)}$$

replace h_n by $R^* \underbrace{h_n}_{\text{onb}} =: f_n \leftarrow$ frame.

~~$\|A\|$~~ $A\|h\|^2 \leq \sum |(h|f_n)|^2 \leq B\|h\|^2$.

M frame-$\textcircled{R}$-bdd if there exist R.

smooth.

$$\sup_{m\in M} \sum |\langle R(m), h_n\rangle_H| < \infty.$$

$$|M|_1 = \inf_{(L,R)} \|L\| \sup \sum |\langle Rm, h_n\rangle|.$$

<u>Lemm</u>. A bdd H^∞, $F \in H^\infty(\Theta; H)$ s.t.

$F(\Theta)$ frame ℓ_1-bdd. $\Rightarrow$ $\Phi_\gamma(F)$ is bdd.

$\underline{\text{Th}^m}$. (H., Haase). X cotype, $K = L^2(\Omega)$, H Hilbert.

$f, g \in H^\infty(\Theta; K)$, $m \in L^\infty(\Omega, H)$.

$$n(z) = \int_\Omega m(t) f(t,z) g(t,z)\, dt \in H^\infty(\Theta; H).$$

If $g(\cdot, A)x \in \gamma(K; X)$, $f(t, A')x' \in \gamma(K, X)$,

then.

$$\Phi_\gamma(n): \begin{cases} X \longrightarrow \gamma(H, X) \\ x \longmapsto (h \mapsto (h|n(z))(A)x. \end{cases} \text{ bdd.}$$