Rafe Lecture 9. 17/10/2012.

P diff. operator of order m.

$Pu = f$ $\qquad \gamma u = \begin{pmatrix} \gamma_0 u \\ \vdots \\ \gamma_{m-1} u \end{pmatrix}$ on Ω smooth mfld w/ bdry.

$B(\gamma u) = h$

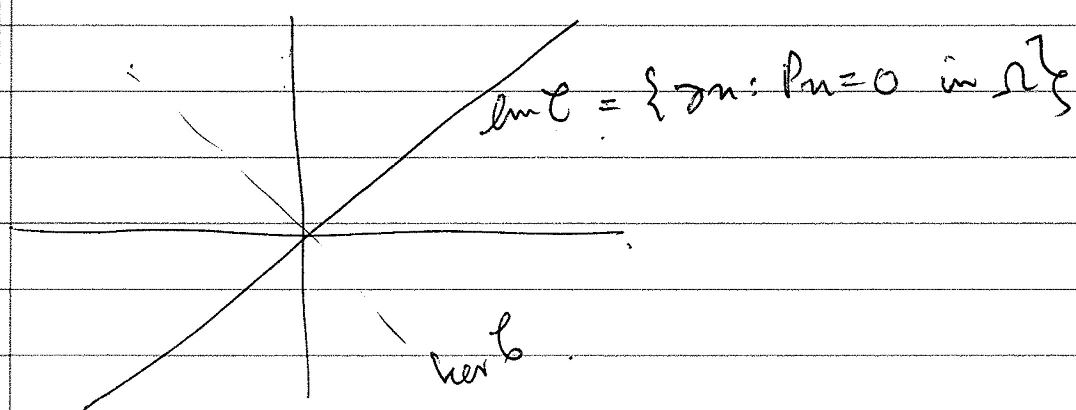
- Extend to $\tilde{P}$ and $\hat{\Omega}$.
- Solve extended "interior" problem.

$\tilde{P}\tilde{u} = \tilde{f}$. $\qquad$ add to v a sol^{n} w of

$B(\gamma v) = K \leftarrow$ mess this $\qquad$ $Pw = 0$.

$B(\gamma w) = h - k$.

$\bigoplus_{j\geq 0}^{m-1} \mathcal{D}'(\partial\Omega)$. $\quad \mathcal{C}$ - Calderon projector.



Work with $A \in \Psi^{\mu}(\partial\Omega)$.

$$\sigma(A)(x', \xi') \sim \sum a_{\mu-j}(x', \xi').$$

s.t. $a_{\mu-j}(x', \xi')$ homogeneous in ξ', $|\xi| \geq 1$.

~~[?]~~

Ex. P elliptic and differential. Then $G \in \mathcal{F}^{-m}$ s.t. $G \circ P \sim I$, $P \circ G \sim I$ has this property!

$$\sigma(G) \sim \sum g_{-m-j}(x,\xi)$$

$$g_{-m} = \frac{1}{P_m(x,\xi)}.$$

This is a very strong hypothesis but this is preserved under coordinate transform and enough for the operators which we are interested.

$u \in C^\infty(\bar{\Omega})$, denote by $u^0 = \begin{cases} u & \text{in } \Omega \\ 0 & \text{outside} \end{cases}$.

$Pu = f$, G parametrix for P.

$$P(u^0) = \underset{\substack{\uparrow \\ \text{classical} \\ \text{int. domain}}}{(Pu)^0} + \tilde{p}(\gamma u).$$

jumps on boundary.

If $P = \sum P_j(x, D_{x'}) D_{x_n}^j$, then

$$\tilde{P} v = \frac{1}{i} \sum_{l+h+1 \le m} P_{l+h+1} v_l \otimes D_{x_n}^h \delta.$$

$u^0 = u \cdot H(x_n)$. H heavyside function positive in x_n direction.

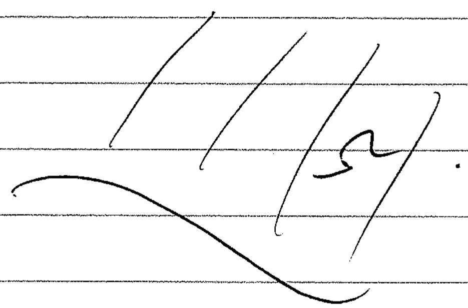
$$D_{x_n}(Hu) = \delta \cdot u + H(D_{x_n} u)$$
$$= u_0 \delta(x_n).$$

$$D^2_{x_n}(Hu) = H D^2_{x_n} u + 2\delta D_{x_n} u + u D^2_{x_n} \delta.$$

$\Delta = D^2_{x_n} + \Delta_y$, then $\tilde{\Delta} = \tilde{D}^2_{x_n}$

For Δ expect. $\frac{1}{i}(v_0 D\delta + v_1 \delta)$.

Apply G. ~~So~~ $u^0 = Gf^0 + G(\tilde{P}(\gamma u))$.



So, this is smooth, because $\tilde{P}(\gamma u)$ supported on bdry, G makes it smooth, and Gf^0 smooth. **But:**

Thm. $\gamma_j u$ makes sense $j = 0, \dots, m-1$

$$u \in C^{m-1}(\overline{\mathbb{R}_+}, \mathcal{D}'(\partial\Omega)).$$

Problem: on a mfld, we can ask for a distribution to ~~have~~ ~~go~~ be good in the normal direction in a coordinate chart. But how do we know that ~~two~~ such guys also are regular (if one has good regularity in another chart).

$Pu^\circ = (Pu)^\circ + \tilde{P}(\gamma u)$.

$u|_{\bar\Omega} = G(f^\circ)|_{\bar\Omega} + G\tilde{P}(\gamma u)|_{\bar\Omega}$

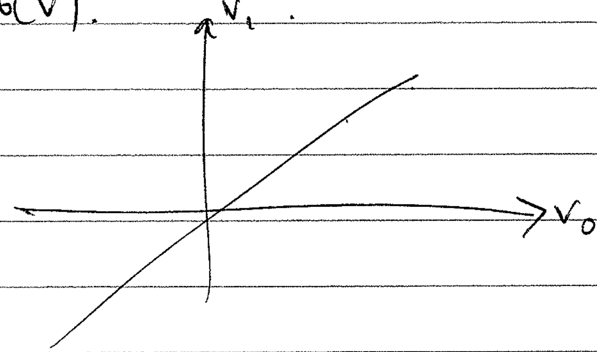
$\gamma_j u = \gamma_j\left(G(f^\circ)|_{\bar\Omega}\right) + \gamma^j\left(G\tilde{P}(\gamma u)|_{\bar\Omega}\right)$.

If $Pu = 0$ in $\Omega \Rightarrow \gamma u = \gamma\left(G\tilde{P}\gamma u\right)|_{\bar\Omega}\Big)$.

That is, $v = (v_0, \dots, v_{m-1})$

$$v \longrightarrow \underbrace{\gamma\left(G(\tilde{P}v)\big|_{\bar\Omega}\right)}_{\mathcal{C}(v)}.$$

This formula is saying $v = \gamma u$, where $Pu = 0$, then $v = \mathcal{C}(v)$.



Calderón's idea:

in Ω $Pu = f$ $\rightsquigarrow$ $(id - \mathcal{C})(\gamma u)$

in $\partial\Omega$ $B(\gamma u) = h$ $\quad = \gamma_j\left(G(f^\circ)|_{\bar\Omega}\right)$.

Then $B(\gamma u) = h$ in $\partial\Omega$

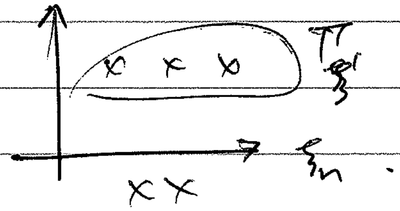
Def. (P, B) is an elliptic bdry problem.
$\Leftrightarrow$ $\begin{pmatrix} I-C \\ B \end{pmatrix}$ is elliptic on $\partial\Omega$.

Need to study $r(G(P_V)|_{\bar\Omega})$. Let $A \in \Psi^{\mu}(\mathbb{R}^n)$ be as above (w/ rationality condition). For $v \in C^\infty(\partial\Omega)$, look at $r_0 A(v\otimes\delta)$ (typically
$$A = D_{x_n}^{j} G\, P_{\ell+k+1} D_{x_n}^{k}, \text{ of order } \mu = j-m+(m-\ell-k-1)+k = j-\ell-1.$$

Thm. $C^\infty(\partial\Omega) \ni v \Rightarrow A(v\otimes\delta) \in C^\infty(\bar\Omega)$.

$A_0 v = r_0(A(v\otimes\delta)) \Rightarrow A_0 \in \Psi^{\mu+1}(\partial\Omega)$, and
$$\sigma_{\mu+1}(A_0) = \frac{1}{2\pi i}\int_{\Gamma_{\xi'}} a_\mu(x', 0, \xi', \xi_n)\, d\xi_n.$$

Here, $\Gamma_{\xi'}$: x', ξ' fixed smooth parameters.
$\rightsquigarrow$ $a_\mu(\xi_n)$ rational, Γ_{ξ} encircles all poles in upper half plane.



For Δ: $P_0 = 0$, $P_1 = 0$, $P_2 = 1$.
Look at $D_{x_n}^{j} G\, D_{x_n}^{1-\ell}$ of order $j-\ell-1$.

$C = (c_{j\ell}) \rightsquigarrow C\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} c_{00} v_0 + c_{01} v_1 \\ c_{10} v_0 + c_{11} v_1 \end{pmatrix}$.
where $c_{j\ell} = r_0(D_{x_n}^{j} G\, D_{x_n}^{1-\ell})$.

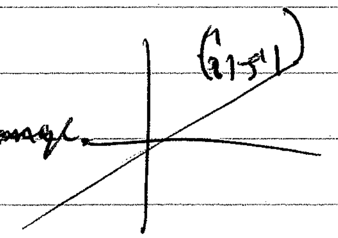
Now, $\sigma_{-2}(G) = \frac{1}{\xi_n^2 + |\xi'|^2} = \frac{1}{2\xi_n}\left(\frac{1}{\xi_n - i|\xi'|} + \frac{1}{\xi_n + i|\xi'|}\right)$

(invariant mean metric on levels $\xi_n = \text{const}$).

Compute $\int_\Pi \xi_n^{j-l+1} \left(\xi_n^2 + |\xi'|^2\right)^{-1} d\xi_n$.

$$= \oint \frac{1}{2} \cdot \frac{\xi_n^{j-l}}{\xi_n - i|\xi'|} \, d\xi_n = 2\pi i \, \frac{1}{2} \left(i\,|\xi'|\right)^{j-l}.$$

$$\Rightarrow \quad C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i|\xi'|} \\ \frac{i}{2}|\xi'| & \frac{1}{2} \end{pmatrix}, \quad \text{with ~~range~~}$$



$\mathbb{C}$.

Why exactly this range? Consider PDE, F-T'd in x'-variables: $(D_{x_n}^2 + |\xi'|^2)u = 0 \rightsquigarrow e^{\pm|\xi'|x_n}$. Take "$-$" (exp. decaying), its Cauchy trace is $\binom{1}{i|\xi'|}$. Kernel of C = Cauchy trace of data good in $\mathbb{R}^-$ but bad in $\mathbb{R}^+$.

• Now, want to prove the theorem. Let $u(x) = v(x')\,\delta(x_n)$. Consider $\langle Au, \Phi \rangle = \int e^{ix\xi} a(x,\xi)\, \hat{v}(\xi')\, d\xi'\, d\xi_n\, \Phi(x', x_n)\, dx'\, dx_n$.

Throw away $|\xi| \le 1$, get

$$\int_{|\xi| \ge 1} \hat{u}(\xi)\, F(\xi)\, d\xi \quad \text{with}$$

$$F(\xi) = \int e^{ix\xi} a(x,\xi)\, \Phi(x)\, dx.$$

(Note: enough to consider homogeneous a since if a is of any lower order $\Rightarrow Au$ is again very smooth, i.e. can't just throw away lower order terms in expansion).

Since $\xi^\alpha F(\xi) = \int e^{ix\xi}(-D_x)^\alpha (a(x,\xi)\Phi(x))\,dx$,
F decays rapidly and so as $e^{ix_n(\xi_n + i\eta_n)} = e^{ix_n\xi_n - x_n\eta_n}$.
$\rightarrow$ F decays rapidly in $\operatorname{Im}\xi_n \geq 0$, provided $\operatorname{spt}\Phi \subset [0,\infty)$.

Next write (using $\hat{u}$ = pol growth, F = decaying in $\operatorname{Im}\xi_n \geq 0$)

$$\langle Au, \Phi\rangle = \int_{\mathbb{R}^{n-1}}\left(\int_{\Gamma_{\xi'}} \hat{u}(\xi', \xi_n) F(\xi', \xi_n)\, d\xi_n\right) d\xi'.$$

Can choose $\Gamma_{\xi'} \sim |\xi'|\Gamma$ (check: poles are hom. func. in ξ').

Choose $\Phi(x) = \varphi(x')\psi(x_n)$, $\operatorname{spt}\psi \subset (0,\infty)$.

$$\Rightarrow \langle Au, \varphi\otimes\psi\rangle = \int_{\mathbb{R}^{n-1}}\int_{\Gamma_{\xi'}} \hat{u}(\xi', \xi_n)\int e^{ix\xi} a(x,\xi)\,\varphi(x')\psi(x_n)\,dx'\,dx_n\,d\xi_n\,d\xi'$$

$$= \int \psi(x_n)\cdot\int_{\mathbb{R}^{n-1}} H(x_n, \xi')\,d\xi'\,dx_n.$$

where $H(x_n, \xi') = \int_{\Gamma_{\xi'}}\int \hat{v}(\xi')\, a(x,\xi)\, e^{ix\xi}\varphi(x')\,dx'\,d\xi_n$.

$$= \int_{\mathbb{R}^{n-1}} \varphi(x') e^{ix'\xi'}\Bigg(\underbrace{\int_{\Gamma_{\xi'}} e^{ix_n\xi_n} a(x', x_n, \xi', \xi_n)\,d\xi_n}_{k(x,\xi')}\Bigg)\hat{v}(\xi')\,d\xi'\,dx'.$$

$$= \int_{\mathbb{R}^{n-1}} \varphi(y') e^{ix'\xi'} k(x, \xi') \hat{v}(\xi')\, d\xi' dx'.$$

Left to do: $k(x', x_n, \xi') = \int_{\Gamma_{\xi'}} e^{ix_n \xi_n} a(x, \xi)\, d\xi_n \in C^\infty([0,\infty)_{x_n} \times S^{\mu+1}_{x', \xi'})$.

Note: $k(x', x_n, \lambda\xi') = \lambda^{\mu+1} k(x', \lambda x_n, \xi')$ by μ-hom of a. Done. Namely, this cpt.

$$\langle Au, \Phi \rangle = \int \underbrace{\overbrace{\langle A_0(x_n) v, \varphi(x') \rangle}^{\text{can restrict to } x_n = 0}}_{\text{smooth}} \psi(x_n)\, dx_n.$$