Rafe Mazzeo Lecture 4. 01/10/2012.

- $K_A(x,y) \rightsquigarrow A \leftrightarrow Op(a)$, $a(x,y,\xi) \in S^m$.

$\mathcal{D}'(\Omega\times\Omega)$ Ψ^m (negligible $\Psi^{-\infty}$)

(negligible smooth functions)

- K_A has stable regularity along diag $\subset \Omega\times\Omega$.

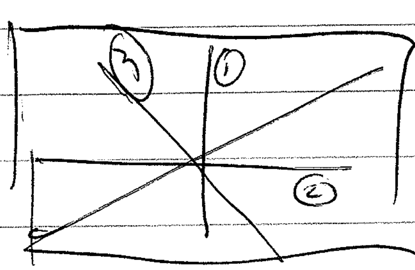
$\exists$ function space E, $V_1, \dots, V_\ell K_A \in E$, $\forall \ell$.
$V_j \in C^\infty$ v.f. tangent to diagonal.

Call such tangent v. fields $\mathcal{V}_b(\Omega\times\Omega, \text{diag})$.

__Claim:__ $C^\infty\text{-span}\{(x_j - y_j)\partial_{x_\ell}, (x_j - y_j)\partial_{y_\ell}\}_{\ell=1}^{n} = \mathcal{V}_b(\Omega\times\Omega, \text{diag})$

__Problem.__ How do you work with this coord-indep. definition?
Ie, $A, B \in \Psi^m, \Psi^{m'} \Rightarrow A\circ B \in \Psi^{m+m'}$.
No hope for doing this ~~computationally~~ via computation. $\rightsquigarrow$ This is where symbols come in. $\rightarrow$ Need to pick E.

Use FT. 3 ways of doing FT. transverse to diag.



(3) - Weyl F.T.

Mainly going to be talking about ① and ②.

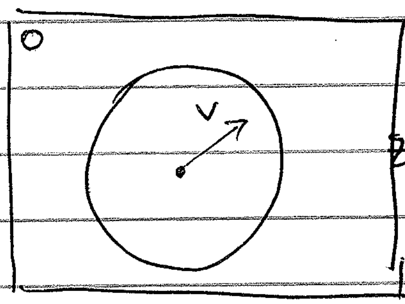
$(x,y) \to (x, y-x)$. (or $(x,y) \to (x-y, y)$).

↑ coordinate sys. ↑ new coord sys.

$$\chi(y-x)\, K_A(x, y-x), \quad \chi \in C_0^\infty(\mathbb{R}^n),\ \chi(0)=1.$$

$$\int \underbrace{\chi(y-x)\, K_A(x, y-x)}\, e^{-i\xi(y-x)}\, dy.$$

cptly spted in $z = y-x$.



$z = y-x$.
v tangent to z.
What are v? $\boxed{z_i \partial_{z_j}}$

So, $$a(x,\xi) = \int e^{-iz\cdot\xi}\, \chi(z) K_A(x,z)\, dz.$$

Now stable regularity means. $z_i \partial_{z_j}(\chi(z) K_A(x,z))$ stays in the same function space. On the side of symbol.

$$(\partial_{\xi_i} \xi_j)\, a(x,\xi) = \int e^{iz\cdot\xi}\, z_i \partial_{z_j} (\chi(z)\, K_A(x,z)).$$

So, condition on operator is

$$|a(x,\xi)| \le C(1+|\xi|)^m.$$

So, $|\partial_\xi^\beta \xi^\alpha a| \le C_{\alpha\beta}(1+|\xi|)^m$.

$$\forall \alpha,\beta,\quad |\alpha|=|\beta| \iff |\partial_\xi^\beta a| \le C(1+|\xi|)^{m-|\beta|}.$$

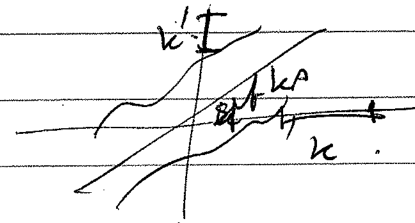
- ~~So, we use F~~
- So, ~~k_A~~ Stable regularity is coord indep in k_A side. But we need to look in coordinates on symbol side. But this is also coord indep. However: how the symbols through change of coordinates relate.

1) Given any $A \in \Psi^m$, $\exists!\, a(x,\xi)$ s.t.

$$A = op(a) + R, \quad R \in \Psi^{-\infty}.$$

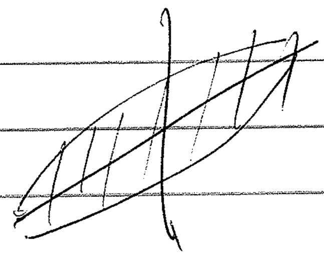
2) very useful to consider operators A which are "properly supported"

spt $(k_A) \subset \Omega \times \Omega$.



$K' = \{ y : \exists (x,y) \in \text{spt } k_A,\ x \in K \}$ compact when K cpt.

Example:



Given $a(x,\xi) \in S^m$, $\exists R \in \Psi^{-\infty}$ s.t.
$op(a) + R$ is properly spted.

3) $A_1 \sim A_2$ if $A_1 - A_2$ is residual ($\in \Psi^{-\infty}$).

$A_1 = Op(a(x,y,\xi))$ and $a(x,y,\xi) \equiv 0$ along diag
then $A_1 \sim 0$.

If $a_1(x,y,\xi) \in S^m$ vanishes to order k on diag.
Then. $A_1 \sim A_2 = op(a_2)$, $a_2 \in S^{m-k}$.

$$a_1(x,y;\xi) = \sum (x_j - y_j)^\alpha$$

4) If A differential operator. Then $spt(Au) \subset spt(u)$.

If $A \in \Psi^*$, then sing spt $(Au) \subset$ sing spt (u).

$u \in \mathcal{D}'(\Omega)$, $\Omega \setminus$ sing spt$(u) = \{x \in \Omega : \exists U \ni x$ and
$f \in C^\infty(U)$ s.t. $\langle u, \varphi \rangle = \int f\varphi$.
$\forall \varphi \in C_c^\infty(U)\}$

distribution.
is smooth at
x.

Pf. Choose $\chi \in C_c^\infty(\Omega)$, s.t. $\operatorname{spt}\chi \subseteq \operatorname{sing\,spt}(u) + B_\varepsilon(0)$.

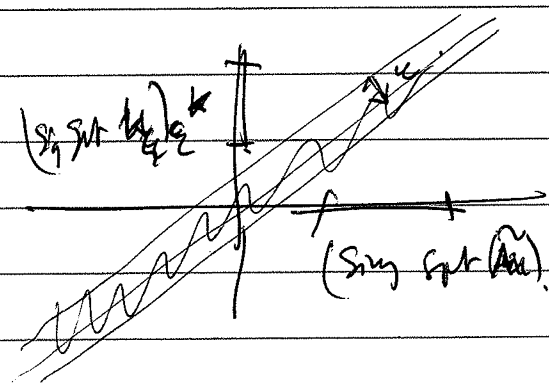
$u = \chi u + (1-\chi)u$, $(1-\chi) \in C^\infty$.

1) $A((1-\chi)u) \in C^\infty$.

2) $A \sim \tilde{A}$, ~~properly~~ properly spted with spt $K_{\tilde{A}} \subset$ ε-nbh of diag.

$(A-\tilde{A})(\chi u) \in C^\infty$.

Left with $\tilde{A}(\chi u)$.



5). $a(x,\xi) \in S^m$, $b(x,\xi) \in S^{m'}$; Choose $R_1 \in \Psi^{-\infty}$, $R_2 \in \Psi^{-\infty}$ so that:

$$A = op(a) + R_1, \quad B = op(b) + R_2 \quad \text{properly spted.}$$

Then $A \circ B = op(c)$, $c(x,\xi) \in S^{m+m'}$.

$$c(x,\xi) \sim \sum \frac{1}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi).$$

$$= ab + \sum D_{\xi_j} a\, D_{x_j} b + \dots$$

$A \circ B - B \circ A \in \Psi^{m+m'-1}$

$$\sigma_{m+m'-1}([A,B]) = \sum_{j=1}^n \left(D_{\xi_j} a\, D_{x_j} b - D_{\xi_j} b\, D_{x_j} a \right) = i\{a,b\}.$$

(Poisson bracket)

Last time we proved: $A \in \Psi^0$, then $A: L^2 \to L^2$ bdd.

$$A^*A + B^*B \sim M^2 I.$$

$$\|Au\|^2 \leq \|M^2 u\| + \iint R(x,y)u(x)u(y)\,dx\,dy, \quad R \in C^\infty(\Omega\times\Omega)$$
$$\leq C\|u\|^2.$$

$L^2(\Omega) \sim$ same w.r.t. any C^∞ (or C^0 metric).

Def^n. $\lambda_s(x,\xi) = (1+|\xi|^2)^{s/2}$

$$op(\lambda_s) = \Lambda_s = (1+\Delta)^{s/2}.$$

Def^n. $H^s(\Omega) = \{\Lambda_{-s}u : u \in L^2\}$.

Note: $A \in \Psi^m$, then $A: H^s \to H^{s-m}$ $\forall s$.

Pf. $L^2 \xrightarrow{\Lambda_{-s}} H^s \xrightarrow{A} H^{s-m} \xrightarrow{\Lambda_{s-m}} L^2$.

$\Leftrightarrow$ $\Lambda_{s-m} \circ A \circ \Lambda_{-s} : L^2 \xrightarrow{bdd} L^2$

- Elliptic regularity is a simple consequence.

$A \in \Psi^m$ elliptic, $A \sim op(a)$, $a(x,\xi) \in S^m$.
s.t. $|a(x,\xi)| \geq C_1(1+|\xi|)^m$, $|\xi| \geq R$.

Thm. If A is elliptic $\Rightarrow \exists B \in \Psi^{-m}$ s.t.
$A \circ B \sim I$, $B \circ A \sim I$.

Pf. $A = op(a)$, find $B_{-m} = op(b_{-m})$ s.t.
$A \circ B_{-m} - I \in \Psi^{-1}$.

$$\sigma_0(A \circ B_{-m} - I) = a\, b_{-m} - 1 \mod S^{-1}.$$

Want $\sigma_0 = 0$. i.e., $b_{-m} = \dfrac{\chi(x,\xi)}{a(x,\xi)}$, $\chi = \begin{cases} 1 & |\xi| \geq R+1 \\ 0 & |\xi| \leq R \end{cases}$.

Set $A \circ B_{-m} - I = R_{-1}$,

$$\sigma_0(R_{-1}) = 0, \quad \sigma_{-1}(R_{-1}) \in S^{-1}.$$

$$\sigma_1(A \circ (B_{-m} + B_{-m-1}) - I - R_{-1}).$$

$$\text{Choose } b_{-m-1} = \frac{\chi\, \sigma_1(R_{-1})}{a(x,\xi)}.$$

Then $\sigma_1 = 0$! Induct to find them.

Set. B_{-m-j} and $\therefore$ $A \circ \underbrace{(B_{-m} + B_{-m-1} + \ldots)}_{B} = I + R$, $R \in \Psi^{-\infty}$.

Now, we can do the same on left to find $B' \circ A \sim I$. But we are only looking upto mod residuals so,

$B' \circ A \circ B \sim B' \circ I \sim I \circ B$! ☐

Rk What have we done? Already seen.

· $A \in \Psi^m$, $u \in H^s \Rightarrow Au \in H^{s-m}$.

<u>Prop</u>. A elliptic, $Au \in H^{s-m} \Rightarrow u \in H^s$.

<u>Pf</u>. $Au = f \in H^{s-m}$.

$$u + Ru = B(Au) = Bf \in H^s. \quad ☐$$

$$\|u\|_s \le \|Bf\|_s + \|Ru\|_s$$
$$\lesssim C(\|f\|_{s-m} + \|u\|_0).$$

Schauder theory – check ops are bdd on function spaces (Hölder) and same business here gives Schauder estimates!

<u>Next time</u>: Some comments/statements for

$$\Box = D_t^2 - \sum_{j=1}^n D_{x_j}^2.$$

$$\sigma_2(\Box) = \tau^2 - |\xi|^2.$$

$\Box u = f$ then gives a precise measure of singularities.