Rafe Mazzeo Lecture 8. 28/09/2012.

- $\Omega \times \mathbb{R}^N$ — Conical structure — Smooth multiplication in $\mathbb{R}^N$ direction.

$$\varphi(x, t\theta) = t\varphi(x,\theta), \quad t \geq 1, \; |\theta| \geq 1$$

$$d_{x,\theta}\varphi \neq 0.$$

$$a(x,\theta) \in S^m: \quad |D_x^\alpha D_\theta^\beta a(x,\theta)| \leq C_{K\alpha\beta} (1+|\theta|)^{m-|\beta|}, \; x \in K \subset \Omega$$

$$I_{\varphi,a} \in \mathcal{D}'(\Omega), \quad u \in C_0^\infty.$$

$$\langle I_{\varphi,a}, u\rangle := \int e^{i\varphi(x,\theta)} a(x,\theta) u(x)\, dx\, d\theta.$$

$$= \int e^{i\varphi(x,\theta)} \underset{\text{transpose}}{(L^t)^k} \big(a(x,\theta)u(x)\big)\, dx\, d\theta. \qquad m-k < -N-1.$$

$$L = \frac{1 + \frac{1}{i}\sum (\partial_{x_j}\varphi)\partial_{x_j} + \frac{1}{i}\sum |\theta|^2 \partial_{\theta_\ell}\varphi}{1 + |\partial_x\varphi|^2 + |\theta|^2|\partial_\theta\varphi|^2}.$$

$$= e + \sum a_j \partial_{x_j} + \sum b_\ell \partial_{\theta_\ell}. \quad a_j \in S^{-1}, \; b_\ell \in S^0, \; e \in S^{-2}.$$

$$Le^{i\varphi} = e^{i\varphi}.$$

$$L^t\big(u(x)a(x,\theta)\big).$$

$$= \underset{S^{m-2}}{e\,a u} + \sum -\underbrace{\partial_{x_j}(a_j \cdot u \cdot a)}_{S^{m-1}} - \sum \underbrace{\partial_{\theta_\ell}(b_\ell\, a(x,\theta) u)}_{S^{m-1}}$$

$|\langle I_{\varphi,a}, u\rangle| \leq C\|u\|_{C^k}$.

In fact $\mathcal{S} = \{x ;\ \exists (x,\theta)\ \partial_\theta \varphi(x,\theta) = 0\}$.

<u>Claim</u>: $I_{\varphi,a} \in C^\infty(\Omega \setminus \mathcal{S})$ ← obstruction to smoothness is exactly $\mathcal{S}$!.

Consider $x \notin \mathcal{S}$.

$$\int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta = \int M^k(e^{i\varphi})\, a$$

$$= \int e^{i\varphi} \left((M^t)^k a\right) d\theta.$$

$$Me^{i\varphi} = e^{i\varphi}, \quad M = \frac{1 + \sum \frac{|\theta|^2}{i} \frac{\partial\varphi}{\partial\theta_\ell}}{1 + |\partial_\theta\varphi|\,|\theta|^2}.$$

* Even though the denominator is $1+$, and nothing blows-up when $\partial_\theta\varphi = 0$, we don't gain anything in singular regularity

* $\mathcal{S}$ is where oscillation ceases. When there's oscillation, we have regularity.

$$\int e^{i(x-y)\xi} a(x,y,\xi)\, d\xi \in C^\infty(\Omega\times\Omega \setminus \text{diag}).$$

$$(x_j - y_j)\, \partial_{x_\ell} \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, d\xi.$$

$$= \int (x_j - y_j)\, \xi_\ell\, e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, d\xi$$

$$= \int \partial_{\xi_j}\, \xi_\ell\, e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, d\xi$$

$$= \int e^{i(x-y)\xi} \underbrace{\left(\xi_\ell\, \partial_{\xi_\ell} a\right)}_{\in S^m}.$$

$$a(x,x,\xi) = 0 \quad \forall x \in \Omega.$$

$$(*) \int e^{i(x-y)\cdot\xi} \underbrace{a(x,y,\xi)}\, d\xi.$$

$$a(x,x,\xi) = \sum a_j(x,y,\xi) \underbrace{(x_j - y_j)}_{\text{interpret as } \partial_{\xi_j}}.$$

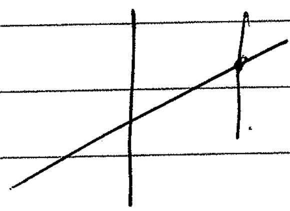
So, $(*) \sim \int e^{i(x-y)\xi} \underbrace{\sum \partial_{\xi_j} a_j(x,y,\xi)}_{S^{m-1}}\, d\xi.$

* Subtle Regularity — if you differentiate tangent to diagonal, regularity is not affected.

* if $a(x,x,\xi) = 0$ then distribution is better than you thought.

1) $|x-y|^{2-n}$. $\quad (x_i-y_i)\partial_{x_\ell}(\text{———})$;
$(\log|x-y|)^N |x-y|^{-\alpha}$; $\quad \delta(x-y)$ etc all have stable regularity.

<u>Stable Regularity</u>: $\cancel{k(x,y)}$ $\quad ((x-y)\partial_x)^\alpha k(x,y)$.



$$\int k(x,y)\, e^{i(x-y)\cdot\eta}\, dy \quad (\text{F.T. in } y).$$

$$\eta^\alpha \partial_\eta^\beta \hat{k}(x,\eta), \quad |\alpha| = |\beta|.$$

(Homogeneous — ie taking ~~F.T.~~ α derivs in F.T. side it gets η^α worse.)

On spatial side this is
$$((x-y)\partial_x)^\alpha k(x,y).$$

$$S^m \ni a \rightsquigarrow A = Op(a).$$

$$(Au)(x) = \int e^{i(x-y)\xi} a(x,y,\xi)\, u(y)\, dy\, d\xi.$$

$$a \in S^m \rightsquigarrow A\ \Psi^m.$$

But going backwards isn't unique. Hmm.

$$0 \longrightarrow \Psi^{m-1} \longrightarrow \Psi^m \longrightarrow S^m/S^{m-1} \longrightarrow 0.$$

Completely coordinate indep symbol.

Operator algebra fact: Ψ^m is highly noncommutative algebra. S^m/S^{m-1} is, however, a commutative representation.

$$\cancel{S^m} \xrightarrow[\text{Quantisation}]{} \Psi^m$$

$$\sigma_{m_1+m_2}(A_1 \circ A_2 - A_2 \circ A_1) = \sigma_{m_1}(A_2)\,\sigma_{m_2}(A_2) - \sigma_{m_2}(A_2)\,\sigma_{m_1}(A_1) = 0$$

$a(x,y,\xi) \rightsquigarrow A = Op(a)$.

$$a(x,y,\xi) \approx \sum \frac{1}{\alpha!} \partial_y^\alpha a(x,y,\xi)\Big|_{y=x} (x-y)^\alpha.$$

$$\underbrace{\int e^{i(x-y)\xi} \partial_y^\alpha a(x,y,\xi)\Big|_{x=y} \frac{(x-y)^\alpha}{\alpha!}}$$

$$\sim \sum \underbrace{a^\alpha(x,\xi)}_{\in S^{m-|\alpha|}}.\,e \qquad \text{(Asymptotic Sum)}.$$

$\boxed{a \to A = op(a)}$

1) $a_j \in S^{m-j}$, $j = 0, 1, 2, \ldots$

Then, $\exists a \in S^m$, $a \sim \sum_{j=1}^{\infty} a_{m-j}$. ← not convergent, but Borel's lemma convergent.

I.e., $a - \sum_{j=0}^{N} a_{m-j} \in S^{m-N-1}$.

2) $Op(a)Op(b) = Op(a\cdot b)$ modulo $\Psi^{m_1+m_2-1}$.
$a \in S^{m_1}$, $b \in S^{m_2}$.

3) $Op(a)^* = Op(a^*)$ mod Ψ^{m-1}.

4) If $A \in S^0$, then $A : L^2_{comp} \to L^2_{loc}$.

Pf

(3) $\langle Op(a)u, v\rangle = \langle u, Op(a)^* v\rangle$

$$\left\langle \int e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi, v(x) \right\rangle$$

$$= \left\langle u(y), \int e^{i(x-y)\xi} \overline{a(x,\xi)} \cdot v(x)\, dx\, d\xi \right\rangle$$

$$a(x,\xi) \sim \sum \frac{\partial^\alpha a(x,\xi)}{\alpha!}\Big|_{x=y} (y-x)^\alpha$$

gives a way to swap x, y.

(2) $A \circ B = A \circ (B^*)^*$. $B^* = Op(b^*)$.

$$\int e^{i(x-y)\xi} a(x,\xi)\, b^*(y,\xi)\, d\xi$$

$$= \int e^{ix\eta} a(x,\eta) e^{-iz\cdot\eta} \left(\int e^{i(z-y)\xi} b^*(y,\xi) \right) d\xi\, dz$$

$$= \int e^{iz(\xi-\eta) - iy\cdot\xi + iz\eta}\, a(x,\xi)\, b^*(y,\eta)\, d\xi\, d\eta\, dz.$$

δ function in integration in z.

$$= \int e^{i(x-y)\cdot\xi} \underbrace{a(x,\xi)\, b^*(y,\xi)}_{s}\, d\xi.$$

$$C(x,\xi) \sim \sum \frac{(\partial_\xi^\alpha a)\, \partial_x^\alpha b(x,\xi)}{\alpha!}$$

4. $A \in \Psi^0$, $A = Op(a)$, $a(x,\xi) \in S^0$ ~~cpt~~ $a \in K$.

$$\Rightarrow |a(x,\xi)| \le M-1, \quad \sqrt{M^2 - |a(x,\xi)|^2} = b_0(x,\xi).$$

Claim: $b_0 \in S^0$.

$$|b_0| \le C, \quad |\partial_\xi b_0| \le \left| \frac{-\partial_\xi |a(x,\xi)|^2}{\sqrt{M^2 - |a(x,\xi)|^2}} \right| \le (1+|\xi|^2)^{-1}.$$

Continue by induction.

Now, $a \in S^0$, $|a| \le M-1$, $b_0 = \sqrt{M^2 - |a|^2} \in S^0$.

$B_0 = Op(b_0)$.

$$\sigma_0(B_0^* B_0) = |\sigma_0(b_0)|^2 = |b_0|^2 = M^2 - |a_0|^2 = \sigma_0(M\, Id - A^* A).$$

$\Rightarrow \sigma_0 (B_0^* B_0 + A^* A - M^2 I) = 0$.

$$B_0^* B_0 + A^* A = M^2 I + R_{-1} \in \Psi^{-1}.$$

Want a -1 order correction term. Find B_{-1} s.t.

$$\sigma_{-1}\big(\underbrace{(B_0+B_{-1})^*(B_0+B_{-1})} + A^* A - M^2 I - R_{-1}\big) = 0.$$

symbol side $0 = b_{-1}^* b_0 + b_0^* b_{-1} + \sigma_{-1}(B_0^* B + A^* A - M^2 I) - \sigma_{-1}(R_{-1})$.

And we find b_{-1} that solves this!

We can successively choose $B_0, B_{-1}, B_{-2}, \ldots$ so they

$$(B_0 + \cdots + B_{-N})^* (B_0 + \cdots + B_{-N}) + A^* A - M^2 Id \in \Psi^{-N-1}.$$

$B \in \Psi^0$, $\quad B^* B + A^* A = M^2 Id + R$, $\quad \underbrace{R \in \Psi^{-\infty}}_{\text{schwartz, very smooth}}$

$\nearrow$ $\|Au\|^2 = \langle Au, Au \rangle = \langle A^* A u, u \rangle$.

Reduce this to

$\leq \langle (A^* A + B^* B) u, u \rangle$.

$\leq \|Au\|^2 + \|Bu\|^2$.

$= M^2 \|u\|^2 + \underbrace{\int R(x,y) u(x) u(y)\, dx\, dy}_{\text{nice!}}$

Smoothing!

History. (*) Riesz transfm. $|x-y|^{2-n}$ type.
operators.

$$\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^{n+2}}.$$

$\partial^{2}_{ij}\Delta^{-1} u = R_{ij} * f.$

(*). Later, ~~pe~~ Bob Seeley in ~~60s~~ 50s?
showed that the Riesz transfms were
coordinate indep. (modulo lower order).
ie → they exist on mfolds!