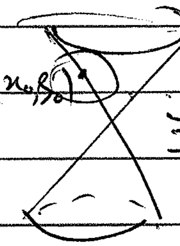
Rauf Lec 17 07/11/2012.

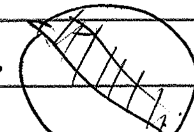
Propagation of singularities.

Idea: $Pu = f$, P is of real principal type.
($p_m = \sigma_m(P)$ is real valued and $dp_m \neq 0$ when $p_m = 0$).



Examples (I) Vacuously, P elliptic
(II) $\Box = P$, $p_2 = \tau^2 - |\xi|^2$.
(III) $P = \frac{1}{i}\frac{\partial}{\partial x}$, $p_1 = \xi_1$.

(x_0,ξ_0) $p_m(x_0,\xi_0)=0$ $\exists U \ni (x_0,\xi_0)$ s.t. $p_m|_U$



Generic model. under change of coords

$\boxed{\text{Assume } m = 1}$

This is ok since we can multiply P by $\Lambda^{1-m}P$.
(Might make diff op $\not p$ a pseudo diff).

"Proof". $P \rightsquigarrow \tilde{P} = \Lambda^{1-m}P$. Then conjugate by F. associated to diff. $\Phi : U \to U$ s.t. $p_1 \circ \Phi = \xi_1$. (Such Φ exists ~~as~~ as change of coordinates).

$$\tilde{\tilde{P}} = F\tilde{P}F^{-1} = \frac{1}{i}\frac{\partial}{\partial x_1} + R \quad \text{in conic nbhd of } (x_0,\xi_0).$$

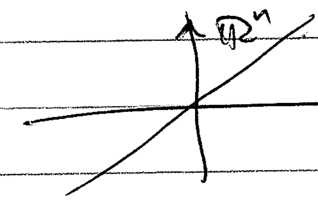
F more general than pseudo diff $\to$ Fourier Integral Operator.

Recall pseudo diff $Pu(x) = \int e^{ix\cdot\xi} a(x,\xi)\, \hat{u}(\xi)\, d\xi$.

FIO : $(Fu)(x) = \int e^{i\varphi(x,y,\theta)} a(x,y,\theta)\, u(y)\, dy\, d\theta$.

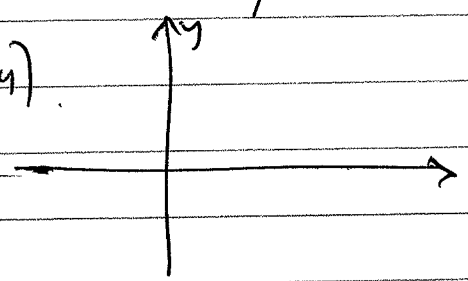
- $\varphi(x,\theta)$ — "phase fn"
- $\varphi \in C^\infty$ on $\mathbb{R}^n \times \mathbb{R}^N$.
- $\varphi(x,t\theta) = t\,\varphi(x,\theta)$, $t \geq 1$.
- $d_{x,\theta}\varphi \neq 0$.
- $a \in S^m$, $|a(x,y,\theta)| \leq (1+|\theta|)^m$.

Recall: If $P \in \Psi^m$, $K_P(x,y)$.



In general, F: fnctn on $\mathbb{R}^k_y$ $\rightarrow$ fn on $\mathbb{R}^l_x$. locally

$K_F(x,y)$.



Cotangent space $\{(x,y,\xi,\eta)\} = \mathbb{R}^{2k+2l}$.

$\cup$

$\Lambda_\varphi \leftarrow$ Lagrangian submanifold.

- No longer sing. set on diagonal, rather, the WF set lives in Λ_φ.

Φ : diff. $\{(x,\xi)\}$; $\underbrace{\text{Graph } \Phi}_{\Lambda} \subset \mathbb{R}^{4n}$.

$e^{it\sqrt{\Delta}}$ operator $W(t)$, $(W(t)u_0)(x) = u(t,x)$.

Solve $(\partial_t^2 - \Delta_x)u = 0$, $u|_{t=0} = u_0$, $u_t|_{t=0} = i\sqrt{\Delta}u_0$.

This is a FIO.

P is of real principal type.

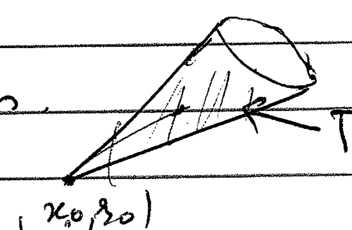
$Pu = f$.

Th^m. $WF(u) = \bigcup$ maximal extended null-bicharacteristics of P_m in $(T^*\mathbb{R}^n)\setminus WF(f)$.

Know: $WF(u) \subset WF(f) \cup \mathrm{Char}(P)$.

$\mathrm{Char}(P) = \{(x,\xi) : P_m(x,\xi) = 0\}$.

(and $(x,\xi) \notin WF(u)$ means



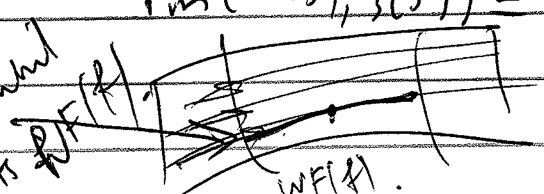
$A \in \Psi^0$.
Γ. ess spt $A \subset \Gamma$.
$a_0(x_0,\xi_0) = 1$.
$Au = h \in C^\infty$.

$P_m(x,\xi) \rightsquigarrow H_{P_m}$ Hamiltonian v. field.

$$H_{P_m} = \sum_{j=1}^n \left(\frac{\partial P_m}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial P_m}{\partial x_j}\frac{\partial}{\partial \xi_j} \right).$$

$(x(s),\xi(s))$ integral curve for H_{P_m}, then,

$P_m(x(s),\xi(s)) \equiv$ const, &



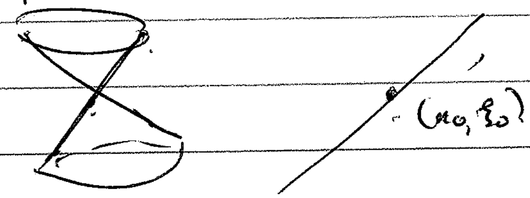
$\{P_m = 0\} = \mathrm{Char}(P)$.

Stays in WF set until int. curve hits WF(f).

WF(f).

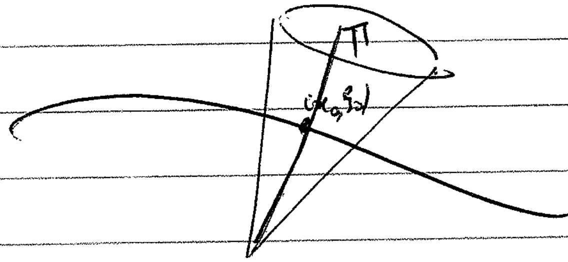
$\frac{1}{i}\frac{\partial}{\partial x}$, $\frac{\partial u}{\partial x_1}=0$, $u = g(x_2,\dots,x_n)$.

$P=\Box$



Pf. (~~Proof~~ of L. Nirenberg '74.)

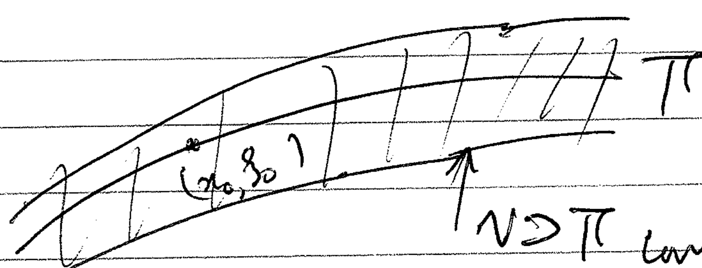
Sps. $f\in C^\infty$. Enough to show that if $(x_0,\xi_0)\notin WF(u)$ and $(x_0,\xi_0)\in \mathrm{char}(P)$.



Can choose W conic nbhd. and an A supported in W s.t. $Au\in C^\infty$, $\sigma_0(A)(x_0,\xi_0)=1$.

Choose $B\in\Psi^0$ ~~s.t. $Bu\in C^\infty$ and~~ s.t. $\sigma_0(B)\neq 0$ along all of Π. s.t. $Bu\in C^\infty$ in nbhd of $\{x_n=(x_n)_0\}$ (plane).

~~To do this~~, and in addition ~~want~~ $[P,B]\in\Psi^{-\infty}$.



$N\supset\Pi$ conical nbhd

$[P,B]\equiv 0$, $B=op(b)$ $b\sim b_0+b_{-1}+b_{-2}+\dots$

What is $\sigma_0([P,B])$. Recall that

$$\sigma(PB) \sim \sum \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha b(x,\xi).$$

$$= p_1 b_0 + \frac{1}{i} \partial_\xi p_1 \partial_x b_0 + \cdots$$

$$\sigma(PB - BP) = \frac{1}{i}\left(\partial_\xi p_1 \partial_x b_0 - \partial_x p_1 \partial_\xi b_0\right)$$

$\sigma_0([P,B]) = H_{p_1} b_0 = 0$ $\rightsquigarrow$ make B constant along ~~bicharacteristic~~ Γ.

$H_{p_1} b_j = \text{junk} \rightsquigarrow b_j|_{(x_0,\xi_0)} = 0$.

$$\left(\begin{array}{l} P(Bu) = \cdots \in C^\infty. \quad \text{why } Bu \in C^\infty \text{ in whole of plane } \{x_n = (x_n)_0\}. \\ \sigma(B(I-A)) = 0 \text{ near } (x_0,\xi_0). \quad (Au \in C^\infty). \\ Bu = BAu + B(I-A)u. \end{array} \right)$$

$$p_m(x,\xi) = (\xi_n - \lambda(x,\xi_1,\ldots,\xi_{n-1}))\underbrace{q(x,\xi)}_{\neq 0}.$$

Reduce to: $L = D_{x_n} - \lambda(x, D_1, \ldots, D_{n-1})$, Q.

$P = LQ \mod \Psi^{-\infty}$.

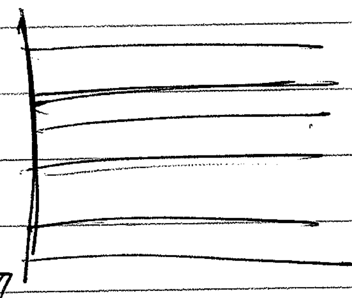
L ← hyperbolic. Q ← elliptic.

$LQBu \in C^\infty$ and $Bu \in C^\infty$ near $x_n = 0$ (for simplicity).

$(D_{x_n} - \lambda)(QBu) = h \in C^\infty$. ODE!

Since Bu smooth near $x_n = 0$, QBu smooth by elliptic regularity and so initial condition smooth.
So, by existence uniqueness, $QBu \in C^\infty$.

<u>Conclusion.</u> $Q(Bu) \in C^\infty$.
$Bu \in C^\infty$ near $x_n = 0$
$\Rightarrow Q(Bu) \in C^\infty$ there



Cauchy data.

Since Q elliptic $\Rightarrow Bu \in C^\infty$. ∎

<u>Egorov's Th^m</u>. $\dfrac{\partial u}{\partial t} = iA(t,x,D)u$. (ie like $\dfrac{\partial u}{\partial t} = i\sqrt{-\Delta}u$).

$A = A_1 + A_0$, A_1 real. ~~$S(t,s)w$~~.

Define $S(t,s)w$ solving
$$\frac{\partial u}{\partial t} = iAu, \quad u|_{t=s} = w \;;\quad S(t,s)w = u|_{t=t}.$$

$P_0 \in \Psi^m$, $P_t = S(t,0)\,P_0\,S(0,t)$ $\quad P_t h$.

flow back to t. $\quad$ flow back from t to 0.

<u>Th^m</u>. $P_t \in \Psi^m$, $\sigma_m(P_t)(x,\xi) = \sigma_m(P_0)(y,\eta)$. where
$$(x,\xi) \xrightarrow{H_{A_1}} (y,\eta).$$

Interesting Corollary.

Let $e^{itA} = S(t,0)$. $e^{itA}u_0 = u(t)$. Then

$WF(u(t)) = C(t)\, WF(u_0)$.

flow by Hamiltonian v-field A_1.