Rafe Lecture 14 29/10/2012.

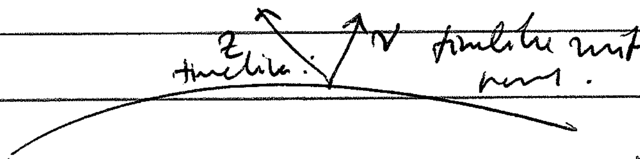
$h = -dt^2 + g$, $\Box u = 0$, $T = du \otimes du + \frac{1}{2}|du|_h^2 \, h$

$x_0 = t, x_1, \dots, x_n$.

$T_{00} = \{ u_t^2 - \frac{1}{2}(-u_t^2 + |d_x u|^2) \}$.

$= \frac{1}{2}(u_t^2 + |d_x u|^2)$.

$T(z, \nu)$

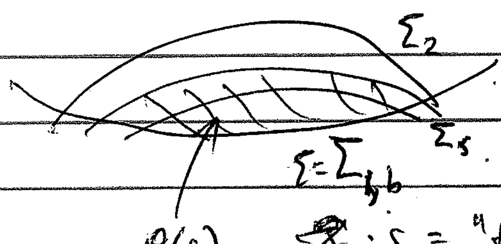


Claim. $T(z,\nu) \simeq (u_t^2 + |d_x u|^2)$.

Pf sketch: Work at a point, assume $\nu = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $h = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$z = (z^0, \dots, z^n)$, $z^0 > 0$, $(z^0)^2 > \sum_{j=1}^n (z^j)^2$.

$T(z,\nu) = z^0 \left(T\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \right) + \sum_{\alpha=1}^n z^\alpha T(\partial_{x_\alpha}, \partial_{x_0})$.



$Lu = f$ $u|_\Sigma = g$

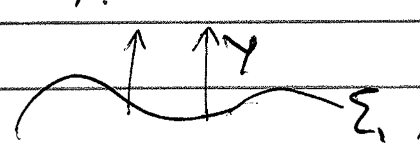
$L = \Box + X$ $du|_\Sigma = \omega$.

$O(s)$, s = "time variable"

$$\int_{\Sigma_2(s)} E_{z,z}(du) \leq C \int_{\Sigma_1} |g|^2 + |\omega|^2 + C \int_{O(s)} |f|^2.$$

$$\|u\|^2_{H^1(O)} \leq C\left(\|Lu\|^2_{L^2(O)} + \|g\|^2_{H^1} + \|g_1\|_{\Sigma_1} \right).$$

$Lu = f$, $u|_{\Sigma_1} = g_0$, $Yu|_{\Sigma_1} = g_1$.



Solve $\left\{\begin{array}{l} Lu = f \\ u|_{S_0} = g_0 \\ Yu|_{S_0} = g_1 \end{array}\right\}$ Reduce to $g_0 = g_1 = 0$

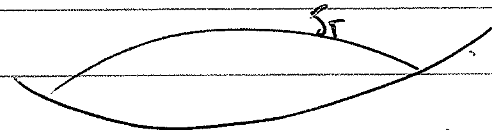
Assume $g_0 \in H^{3/2}$, $g_1 \in H^{1/2}$.

Find $\mathcal{U} \in H^2$ s.t. $\mathcal{U}|_{S_0} = g_0$, $Y\mathcal{U}|_{S_0} = g_1$

$$L(u - \mathcal{U}) = f - L\mathcal{U} \in L^2.$$

$$(u - \mathcal{U})|_{S_0} = 0, \quad Y(u - \mathcal{U})|_{S_0} = 0.$$

$V_T(\mathcal{O}) = \{ w \in C^\infty(\overline{\mathcal{O}}) : w = dw = 0 \text{ on } S_T \}$. $\mathcal{O} = \cancel{\Omega} \cup S_T$.



L^*, $\left(\langle Lu, v \rangle_{\mathcal{M}} = \langle u, L^* v \rangle_{\mathcal{M}} \right)$, $L = \Box + X^*$.

$v \in V_T(\mathcal{O}) \rightsquigarrow \langle f, v \rangle$, fix $f \in L^2$.

$$|\langle f, v \rangle| \leq \|f\|_{L^2(\mathcal{O})} \|v\|_{L^2(\mathcal{O})} \leq \|f\|_{L^2(\mathcal{O})} \|v\|_{H^1(\mathcal{O})}$$
$$\lesssim \|f\|_{L^2(\mathcal{O})} \|L^* v\|_{L^2(\mathcal{O})}.$$

$\langle f, v \rangle_{L^2} = \langle u, L^* v \rangle_{L^2}$, $u \in L^2$ by Riesz Rep.

$\partial_t^2 u - \Delta u = -Xu + f \Rightarrow u \in \mathrm{Lip}(\mathbb{R}, H^{-2}(S_0))$

$\partial_t u \in AC(\mathbb{R}, H^{-2}(S_0))$.

~~Let~~.

Consider $\mathbb{R}_t \times (\mathbb{T}^n_x)$. (for simplicity, and avoid cross terms).

$u_\alpha = D_x^\alpha u$

$$L D^\alpha = D^\alpha L + \sum_{|\beta| \le |\alpha|} X_\beta D_\beta .$$

$$L u_\alpha = \sum_{|\beta| \le |\alpha|} x_\beta u_\beta = D^\alpha f .$$

System of equations for $\{u_\alpha\}_{|\alpha| \le k-1}$, $u_\alpha|_{S_0}$, $Y u_\alpha|_{S_0}$ determined by f, g_0, g_1.

$$\sum_{|\alpha| \le k} \|D^\alpha u\|^2_{H^1(\Theta)} \le C \sum_\alpha \left(\|D^\alpha f\|_{?} + \|g_{0\alpha}\|_{H^1(S_0)} + \|g_{1\alpha}\|^2_{L^2(S_0)} \right).$$

$$\le C \left(\|f\|^2_{H^{k-1}(\Theta)} + \|g_0\|_{H^k(S_0)} + \|g_1\|^2_{H^{k-1}(S_0)} \right).$$

$f \in C^\infty(\Theta)$, $g_0^{(j)}, g_1^{(j)} \in C^\infty(S_0)$, by mollification,

u_j solution to Wave Eq, but smooth, not just C^2.

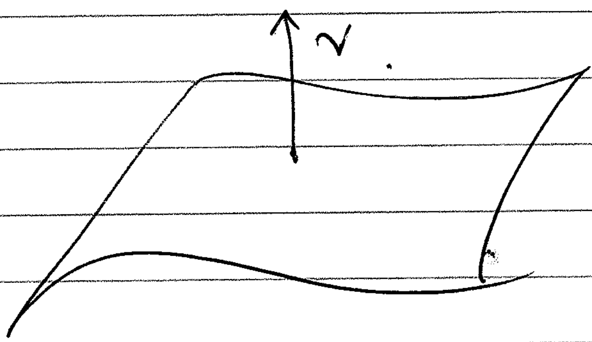
Hyperbolic Systems. $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$.

$$L u = \partial_t u + \sum \underbrace{A_j(t,x)}_{} \frac{\partial u}{\partial x_j} + Bu = f .$$

Assume: $A_j^* = A_j$ Really are matrices. derivatives out front.

$\nu = \begin{pmatrix} \nu_0 \\ \vdots \\ \nu_n \end{pmatrix}$ (normal in ~~t or~~ t-direction).

$$\frac{1}{i}\sigma_2(t,x,\nu) = \nu_0 I + \sum_{j=1}^{n} A_j \nu_j .$$



S spacelike.

$\iff$ $\frac{1}{i}\sigma_2(t,x,\nu)$ pos/neg. $\underline{\text{definite}}$

$$\frac{1}{i}\sigma_2(z,\zeta) = \lambda(z,\zeta)$$

$z = (t,x)$, $\quad \langle Lu, v\rangle_{L^2(\mathcal{O})} - \langle u, L^* v\rangle_{L^2(\mathcal{O})} = \underbrace{\frac{1}{i}\int_{\partial\mathcal{O}} \langle \lambda(z,\nu)u, v\rangle}_{\substack{\text{comparable to } \|u\|^2_{\partial\mathcal{O}} \\ \text{by pos. def. hyp on } A_j}}$.

$\Sigma_{1,b}$.

$$\int_{\Sigma_2(s)} \langle \lambda(z,\nu_2)u,u\rangle \le \int_{\Sigma_{1,b}(s)} \langle \lambda(z,\nu_1)u,u\rangle + k\int_{\mathcal{O}(s)} |u|^2 .$$

$$k\int_{\mathcal{O}(s)} |f|^2 + k\int_{\Sigma_1} |g|^2 .$$

If $E(s) = \int_{\mathcal{O}(s)} \langle \lambda(z,\nu)u,u\rangle$.

$$\Rightarrow E'(s) \le C E(s) + \underbrace{F(s)}_{\text{controlled}}.$$

Gronwall's Inequality.

$$\left(e^{-Cs}E(s)\right)' = e^{-Cs}E' - Ce^{-Cs}E \le e^{-Cs}F(s)$$

and we get

$$\int_{O(s)} |u|^2 \le C(s-s_0)\int_{\Sigma_1} |u|^2 + C\int_{O(s)} |f|^2.$$

$$s_0 \le s \le S.$$

Basic energy estimate. $L^2 \to L^2$ b/c we're not elliptic.

$Lu = f$, $u|_{S_0} = g$, by duality $\|v\|_{L^2(O)} \le C\|L^* v\|_{L^2(O)}$.

Then Riesz rep gives us existence of sol$^{\underline{n}}$s.

$\partial_t u = Au$; $A(t,z,D_z) \in \Psi^1$ (ΨDO).

$A + A^* \in \Psi^0$.

smooth family of Ψ diff ops parametrized by t.

Prob. A. not local.

On $\mathbb{R}^n_z$, take $\varphi(\xi) \in \mathcal{S}$, $\varphi(0) = 1$, $J_\varepsilon = \varphi(\varepsilon D_z)$.

$$\varphi(\varepsilon D_z)u = \mathcal{F}^{-1}\left(\varphi(\varepsilon\xi)\hat{u}(\xi)\right), \quad J_\varepsilon \in \Psi^{-\infty}$$

and $J_\varepsilon \to Id$ in Ψ^0

$$\partial_t u_\varepsilon = J_\varepsilon A J_\varepsilon u_\varepsilon + \tilde{g}. \quad u_\varepsilon|_{t=0} = f$$

Solves exactly as ODE since $J_\varepsilon A J_\varepsilon$ bdd ops.

<u>Claim</u> $K(t) \in \mathcal{B}(\mathcal{H}) \Rightarrow \partial_t v = K(t) v$ solves easily.

$f \in H^s(\mathbb{R}^n)$, $g \in C(\mathbb{R}, H^s(\mathbb{R}^n))$ $s \geq 0$.

Find $u_\varepsilon \in C^1(\mathbb{R}, H^s(\mathbb{R}^n))$ Goal $\|u_\varepsilon(t,\cdot)\|_{H^s} \leq C(t)$ indep of ε.

$$\Lambda^s = (1+\Delta)^{s/2}, \quad \partial_t \|\Lambda^s u_\varepsilon\|_{L^2}^2.$$