Rafe Lec 10.

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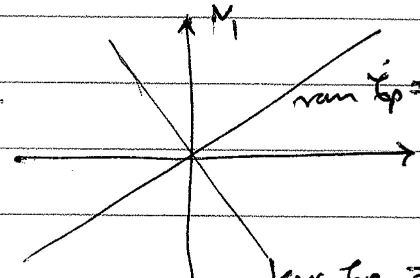
P elliptic, differential of order m.

Calderon projector C_P acting on m-tuples of functions (distributions) on $\partial\Omega$.

Big Picture:



$\exists$ choice of extension. $\left\{ \right.$ (remember we extend P to $\tilde{P}$ by extend Ω to $\tilde{\Omega}$ (cpct). C_P depends on all these.)

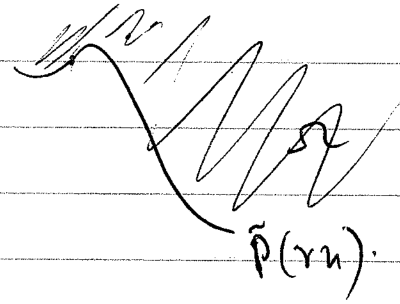
$$\left.\begin{array}{l} Pu = 0 \\ B(\gamma u) = h \end{array}\right\} \text{ solvability} \iff B|_{\text{ran } C_P} \text{ isomorphism}$$

- $u \rightsquigarrow u^0$ = extension by 0 outside Ω.
- $P(u^0) = (Pu)^0 + \tilde{P}(\gamma u)$.

$$\tilde{P}v = \sum_{k+l+1 \le m} P_{l+k+1} \, v_l \otimes D_{x_n}^k \delta(x_n).$$

- Apply G = parametrix for $\tilde{P}$.

$G\circ P \sim I. \Rightarrow u^0 = G(f^0) + G(\tilde{P}\gamma u).$



~~$\gamma u = \gamma$~~ Recall $G(\tilde{P}\gamma u)$ extends smooth upto bdy!

$$\gamma u = \gamma\left(G f^0|_{\bar\Omega}\right) + \gamma\left(G(\tilde{P}\gamma u)|_{\Omega}\right).$$

If $Pu=0$, then $\gamma u = C_P(\gamma u)$ where.

$$C_P v = \gamma\left(G\tilde{P}v|_{\bar\Omega}\right), \quad v = \begin{pmatrix} v_0 \\ \vdots \\ v_{m-1}\end{pmatrix}.$$

$$C_P\begin{pmatrix} v_0 \\ \vdots \\ v_{m-1}\end{pmatrix} = \begin{pmatrix} C_{00} & \cdots & C_{0(m-1)} \\ \vdots & & \\ C_{(m-1)0} & \cdots & C_{(m-1)(m-1)}\end{pmatrix}\begin{pmatrix} v_0 \\ \vdots \\ v_{m-1}\end{pmatrix}.$$

$$C_{jl}\in \Psi^{j-l}(\partial\Omega).$$

$$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}.$$ ← because it is a projector!

$$\tilde{P}v = \sum P_{k+l+1}\, v_l \otimes D^k_{x_n}\delta.$$

$$\Rightarrow C_{jl}v_l = \gamma_0\underbrace{\left(D^j_{x_n} G \sum_{k=0}^{m-l-1} P_{l+k+1} D^k_{x_n}\right)}_{\text{order}\ j-\cancel{m}+\cancel{m}-l-\cancel{k}+\cancel{k}-1\ =\ j-l-1}[v_l\otimes\delta(x_n)].$$

$A \in \mathfrak{F}^{\mu}(\bar{\Omega})$, $\quad v \mapsto r_0\left(A(v \otimes \delta)\right) = A_0 v$.

$\boxed{A_0 \in \mathfrak{F}^{\mu+1}(\partial\Omega)}$ Th:

$\Rightarrow \quad \tau_{j\ell} \in \mathfrak{F}^{j-\ell}(\partial\Omega)$.

<u>Idea:</u> $\quad u(x) = v(x')\,\delta(x_n)$.

$$\langle Au, \Phi \rangle = \cancel{\Phi}\ \cancel{\xi} \int_{|\xi| \geq 1} \hat{u}(\xi)\, F(\xi)\, d\xi .$$

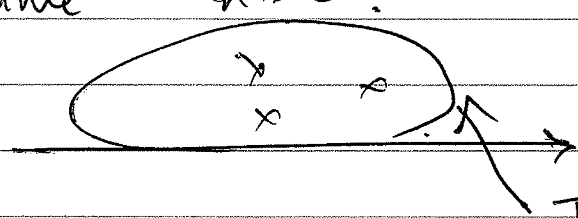
here $\Phi(x) = \varphi(x')\,\psi(x_n)$.

$$F(\xi) = \int e^{ix\xi}\, a(x,\xi)\, \varphi(x)\, dx .$$

decays rapidly. even in the region $\operatorname{Im}\xi_n \geq 0$.

$$i x_n \xi_n \to i x_n(\xi_n + i\eta_n) = i x_n \xi_n - x_n \eta_n$$

since $x_n \geq 0$.



$\xi_n \in \mathbb{C}$.

$\Pi_{\xi'} \simeq |\xi'|\,\Pi_1$.

$a(x,\xi)$ – assumed homogeneous of order μ in $|\xi|$.

Since every operator is ~~homogen~~ a sum of homogeneous terms. <u>not</u> rational.

Eventually,

$$\underbrace{\int_{\mathbb{R}_{\xi'}} a(x', 0, \xi', \xi_n)\, e^{i x_n \xi_n}\, d\xi'_n}_{\text{Symbol of } A_0} \Big|_{x_n = 0}$$

$$\langle Au, \Phi\rangle = \int \langle A_0(x) v, \varphi(x') \rangle \psi(x_n)\, dx'\, dx_n.$$

<u>Question</u>: What do the poles of A mean (in our special case at least)?

$A = D^j_{x_n} G P_{l+k+1} D^k_{x_n}$. $\Rightarrow$ Poles of A come from G.

$P \rightsquigarrow P_m(x, \xi)$ $\qquad \sigma_{-m}(G) = \dfrac{1}{P_m(x,\xi)}$.

$\dfrac{1}{p(x, \xi', \xi_n)}$ $\qquad \xi_n \to P_m(x, \xi', \xi_n)$ <u>polynomial</u>

Ellipticity $\Rightarrow$ No zeros on $\mathbb{R}$.

But polynomial $\Rightarrow \exists$ zeros!

Now something particular to this thing being diff.

$P_m(x, -\xi) = (-1)^m P_m(x, \xi)$.

Prop # zeros in upper half plane = # zeros LHP.
provided dim $n \geq 3$.

If P is elliptic (and scalar) and dim $n \geq 3$.
then deg P = even.

$n=2$ this fails - Cauchy-Riemann operator.

$P(x', 0, \xi', D_{x_n})$ const coeff. ODE on $\mathbb{R}_{x_n}$.

$Pu(x_n) = 0$

FT (or Lap. T) $p(x', 0, \xi', \xi_n)\, \hat{u}(\xi_n) = 0$.

Guess $u(x_n) = e^{i x_n \xi_n} \Rightarrow$ solve $P(x', 0, \xi', \xi_n) = 0$.

Leads us to define $S^{\pm} = \{ u(x', \xi', x_n) : P(x', 0, \xi', D_{x_n}) u = 0$
u tempered in $\mathbb{R}^+$ or $\mathbb{R}^-$ $\}$.

$S^+ \cap S^- = \{0\}$ by ellipticity assumption. $S^+ \Leftrightarrow$ zeros in upper half plane.
S^- zeros in lower.

Also, dim $S^{\pm} = m/2$. for $n \geq 2$.

Let. $D^{\pm}(x', \xi') = \{ \gamma u : u \in S^{\pm} \} \cong S^{\pm}(x', \xi')$.
in $0, \ldots, (m-1)$ order derivatives in x_n — by uniqueness & existence for ODEs.

We get $\mathbb{C}^m = D^+(x', \xi') \oplus D^-(x', \xi')$. by dim count.

Writing $\sigma(C_p)(x', \xi') = (\sigma_{j-k}(C_{jk})(x', \xi'))$, we have

<u>Prop</u>. $\sigma(C_p)$ is the projection onto D^+ along D^- (for any pair $(x', \xi') \in T^*\partial\Omega$).

<u>Pf</u>. Take $u \in S^+$, look at ~~$P(D_{x_n})(x^0) = \frac{1}{i}\Sigma$~~.

$$P(D_{x_n})(x^0) = \frac{1}{i}\Sigma\, P_{l+k+1}(x', \xi')\, \gamma_l u \otimes D^k \delta(x_n).$$

$$\Rightarrow (\quad)^\wedge(\xi_n) = \frac{1}{i}\Sigma\, P_{l+k+1}(x', 0, \xi')\gamma_l u \xi_n^k.$$

$$\Rightarrow \text{For } x_n > 0,\ u(x_n) = \frac{1}{2\pi i}\int e^{ix_n\xi_n} \frac{\sum P_{l+k+1}(x', 0, \xi')\xi_n^k \gamma_l u}{P(x', 0, \xi', \xi_n)}\, d\xi_n$$

$$\text{Thus, } \gamma_j u = \frac{1}{2\pi i}\int e^{ix_n\xi_n} \frac{\sum P_{l+k+1}\xi_n^{j+k}(\gamma_l u)}{p(\xi_n)}\, d\xi_n.$$

But this just says $\gamma u = C(x', \xi')(\gamma u)$.

$\Rightarrow D^+ \subset \operatorname{Im} C^+$, likewise $D^- \subset \operatorname{Im} C^-$

Contour deformation into lower $\frac{1}{2}$ plane.

$C^+ + C^- = Id$. (contour enclosing all poles → contour integral).

$\Rightarrow$ $D^\pm = \operatorname{Im} C^\pm$, has dimension counting.

To finish up $\begin{cases} Pu = f \\ B_r u = h \end{cases} \Leftrightarrow \begin{cases} (I-C)\cdot(\gamma u) = r(G f^0|_{\mathbb{R}^n}) \\ \beta(\gamma u) = h \end{cases}$ (up to smoothing).

<u>Historically</u>: Lopatinski-Schapiro.

$\delta^+_p = \{ u$ "good" soln. on $\mathbb{R}^+ \} \longleftrightarrow D^+_p$.

$B_r (\gamma u) = \sum_{s=0}^{m-1} B_{rs}(x', D_{x'}) \partial_{x_n}^s u|_{\partial_{x_n}} = 0$.

$r = 1, \ldots, q$

$\underset{v \in \mathbb{C}^{mk}}{\gamma u} \longrightarrow \underbrace{\left(\sigma(B_{rs})(x', \xi') \gamma_s u\right)_{r=1}^{q}}$.

$(D^+_p \to \mathbb{C}^{m/2})$ $\qquad$ $q \times m$ matrix, acting on $D^+_p = m/2$-dim.

→ For this to be an isomorphism, need $q = m/2$.

Comparing w/ condition for (P, B) to be ell. bdry problem, $\begin{pmatrix} 1-C \\ B \end{pmatrix}$ elliptic.

→ need $B = m/2 \times m$ matrix since $I - C$ projects onto $m/2$-dim subspace. And β = isomorphism on $\ker(1-C) = \operatorname{Im} C$.

This is precisely Lopatinski-Schapiro.