~~Re~~ Lecture 1 PDE.

Elliptic Equations, Next week: Inverse conductivity Problem.
Rest of quarter: Hyperbolic.

Text: M. Taylor Vol 2,
other refs: Taylor vol I, Chazarain - Piriou, Hörmander
(Unpublished lecture notes : Melrose, Uhlmann.
Shubin - Pseudodiff. operators and spectral theory.
G. Grubb - Distributions and operators

__Schedule__: First 2 weeks - meet MWF.

Some Review of Dist. theory, Fourier transform, Sobolev Spaces.

$\Omega \subset \mathbb{R}^n$, domain

"smallest" function class $C_0^\infty(\Omega) = \{ u$ smooth in Ω, cptly. supported $\}$.

$|D^\alpha u| \le C_\alpha$, $\forall \alpha$ $\quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. $\quad D^\alpha = D^{\alpha_1} \dots D^{\alpha_n}$.

convenient for Fourier transf.

__Topology__: $u_j \to u$ in C_0^∞ if ① $\operatorname{spt} u_j \subset K \subset\subset \Omega$
② $D^\alpha u_j \to D^\alpha u$ in C^0.

$C_0^\infty(\Omega)' =: \mathcal{D}'(\Omega) = \{\text{distributions on } \Omega\}.$

$u \in \mathcal{D}'(\Omega)$ means $\forall K \subset\subset \Omega, \exists N \in \mathbb{N}$ s.t. $\forall \varphi \in C_0^\infty(\Omega)$, supp $\varphi \subset K$,

$\varphi \mapsto \langle u, \varphi \rangle$ satisfies

$$|\langle u,\varphi\rangle| \leq C \sup_{|\alpha| \leq N} \sup_{x \in \Omega} |D^\alpha \varphi(x)|.$$

Heuristically: u only sees a finite number of derivatives on a compact set.

Ex. 1) $u \in L^1_{loc}$. $\langle u, \varphi \rangle = \int u \varphi$.

2) δ_p, $\langle \delta_p, \varphi \rangle = \varphi(p)$.

3) $\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, (D)^\alpha \varphi \rangle$.

Topology on $\mathcal{D}'(\Omega)$ is the weak topology.

FOURIER TRANSFORM

$u \in \mathcal{S}$ means $|x^\beta D^\alpha u| \leq C_{\alpha\beta}$.

$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx$.

as $|\xi| \to \infty$, oscillates faster

$D_{\xi_j}\hat{u}(\xi) = (+x_j u)^\wedge(\xi)$.

$\xi_u e^{-ix\xi} = i\,\partial_{x_u} e^{-ix\xi}$.

$\xi_u \hat{u}(\xi) = (D_{x_u} u)^\wedge(\xi)$.

<u>Point</u>. $\xi^\alpha D_\xi^\beta \hat{u}(\xi) = i^q \cdot (\partial_x^\alpha x^\beta u)^\wedge(\xi)$.

$(F^{-1}\hat{u})(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\,\hat{u}(\xi)\,d\xi$.

So, $F : \mathcal{S} \to \mathcal{S}$ linear isomorphism.

But $(F^{-1}\hat{u})(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi$.

A priori doesn't make sense.

So, this is "defined" by.

$$(2\pi)^{-n}\lim_{\varepsilon\to 0}\iint e^{i(x-y)\cdot\xi - \varepsilon|\xi|^2}\,u(y)\,dy\,d\xi.$$

and then this $= u(x)$.

$(uv)^\wedge = \hat{u} * \hat{v} = \int \hat{u}(\eta)\,\hat{v}(\xi-\eta)\,d\eta$.

$u \in \mathcal{S}'$, $\varphi \in \mathcal{S}, \varphi \mapsto \langle u, \varphi\rangle$

$$|\langle u,\varphi\rangle| \leq C \sum_{|\alpha|+|\beta|\leq N} |x^\alpha D_x^\beta \varphi|$$

~~§ Define F.T. of Tempered distribution,~~

For $u, v \in \mathcal{S}$,

$$\int \hat{u}(\xi) v(\xi) \, d\xi = \int u(x) \hat{v}(x) \, dx.$$

$$= \int e^{-ix\xi} u(x) v(\xi) \, dx \, d\xi.$$

<u>Define</u>:

① $u \in \mathcal{S}'$, $\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle$. ~~(define)~~

② Plancherel: $\int \hat{u} \bar{\hat{u}} = \int |\hat{u}|^2 = (2\pi)^n \int |u|^2 \, dx.$

③ $u \in L^2 \subset \mathcal{S}'$, $u_j \in L^2_{cmp}$. $u_j \xrightarrow{L^2} u$.

$\hat{u}_j$ Cauchy in L^2. defines $\hat{u} \in L^2$.

$\Rightarrow \mathcal{F}: L^2 \xrightarrow{\sim} L^2$.

L^2-based Sobolev Spaces

$$L^2(\mathbb{R}^n) = H^0.$$

<u>Def</u>^{n}. $s \in \mathbb{R}$: $H^s(\mathbb{R}^n) = \{ u : \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty \}$.

it's s, not $s/2$ becaus the F-T is taken.

$u \in L^2$, $D_{x_j} u \in L^2 \; \forall j$.

$$\Rightarrow \int |\hat{u}|^2 + \Sigma_j |\xi_j \hat{u}|^2 < \infty.$$

$$\Rightarrow \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^1 < \infty$$

So, $u, D^\alpha u \in L^2$, $|\alpha| \le k \iff u \in H^k$.

How does these intermediate spaces arise?

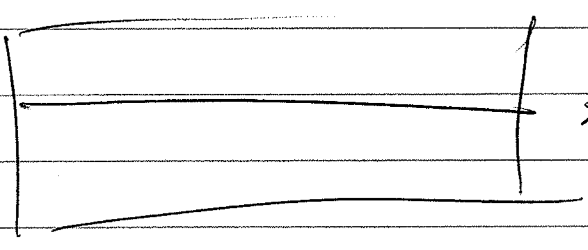
Interpolation. F.T. gives maps b/w endpoints and take endpoints to be integer spaces.

$$H^{-s} = (H^s)', \quad u \in H^s, \; v \in H^{-s}$$

$$\langle u, v\rangle = \int u(x) v(x) = \int \hat{u}(\xi)\hat{v}(\xi).$$

$$\left|\int \hat{u}\hat{v}\right| \le \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \int |\hat{v}(\xi)|^2 (1+|\xi|^2)^{-s}.$$

<u>Prop</u> $u(x', x_n) \longrightarrow u(x', 0)$. extends the $\mathcal{S}$ as a bounded mapping from $H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ if $s > \frac{1}{2}$.



$x_n = 0$

<u>Pf</u> $\hat{u}(\xi', \xi_n) \rightsquigarrow \tilde{u}(\xi', x_n) = \int e^{i x_n \xi_n} \hat{u}(\xi', \xi_n)\, d\xi_n$.

$$\tilde{u}(\xi', 0) = \int \hat{u}(\xi', \xi_n)\, d\xi_n.$$

$$|\hat{u}(\xi',0)|^2 \le \int |\hat{u}(\xi',\xi_n)|^2 (1+|\xi|^2)^s d\xi \cdot \int (1+|\xi|^2)^{-s} d\xi_n$$

$$\int (1+|\xi'|^2 + \xi_n^2)^{-s} d\xi_n.$$

Change of variables: $\xi_n = (1+|\xi'|^2)^{\frac{1}{2}} \eta_n$.

So,
$$\int (1+|\xi'|^2+\xi_n^2)^{-s} d\xi_n$$
$$= (1+|\xi'|^2)^{\frac{1}{2}-s} \underbrace{\int (1+\eta_n^2)^{-s} d\eta_n}_{C_s}.$$

$$\int |\hat{u}(\xi',0)|^2 (1+|\xi'|^2)^{s-\frac{1}{2}} d\xi'$$
$$\le C_s \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s d\xi.$$

$$\|Ru\|_{s-\frac{1}{2}} \le C_s \|u\|_s.$$

__Prop.__ $R: H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, is __onto__.

__Pf.__ Construct extension map. Let $g(x') \in H^{s-\frac{1}{2}}$, $\tilde{g}(\xi')$.

$$\hat{u}(\xi) = \frac{\tilde{g}(\xi')(1+|\xi'|^2)^{s-\frac{1}{2}}}{(1+|\xi|^2)^s}.$$

So,

$$\int |\tilde{g}(\xi')|^2 \frac{(1+|\xi'|^2)^{2s-1}}{(1+|\xi|^2)^{2s}} (1+|\xi|^2)^s \, d\xi$$

$$= \int |\tilde{g}(\xi')|^2 (1+|\xi'|^2)^{2s-1} (1+|\xi'|^2+\xi_n^2)^{-s} \, d\xi_n \, d\xi'$$

$$= C_s \int |\tilde{g}(\xi')|^2 (1+|\xi|^2)^{s-\frac{1}{2}} < \infty.$$

Also, need to chk. $\tilde{u}(\xi', 0) = \int \hat{u}(\xi', \xi_n) \, d\xi_n$
$= C_s' \tilde{g}(\xi')$. $\boxtimes$

1) For any domain $\Omega \subset \mathbb{R}^n$. define ~~$H^s_{loc}(\Omega) = \{u$~~

$$H^s_{loc}(\Omega) = \{u : \varphi u \in H^s(\mathbb{R}^n) \ \forall \varphi \in C_0^\infty(\Omega)\}.$$

2) $f : \Omega \to \Omega'$. $\langle F_* u, \varphi \rangle = \langle u, F^* \varphi \cdot \det DF \rangle$,

$$F^* \varphi = \varphi(F(x))$$

And Diffeomorphism don't screw up sobolev regularity locally.

3) $H^s \subset C^0$ if $s > n/2$, $H^s \subset C^{k,\alpha}$ if $s > n/2 + k + \alpha$.

where we're going:

$$P(x,D) = \sum_{|\alpha|\le m} P_\alpha(x) D^\alpha \quad \text{PDO}$$

$Pu = f$ ← existence?
← regularity.

<u>History</u>: • Even into the middle of last century, people were writing explicit sol^{n}s.

• This is almost 0 real world PDE.
• Better to answer the E, R problem abstractly.
• E - R are different sides of the same coin.

$u \in \mathcal{S}'(\mathbb{R}^n)$.

$$P(x,D)u = \sum P_\alpha(x) D_x^\alpha u(x)$$

$$= \int e^{ix\xi} \Big(\sum_{|\alpha|\le m} P_\alpha(x) \xi^\alpha \Big) \hat{u}(\xi)\, d\xi.$$

But, now consider:

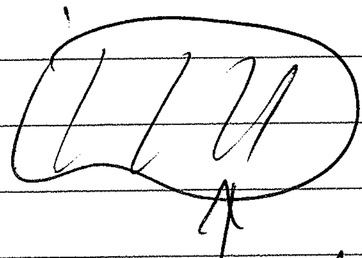
$$A(x,D)u = \int e^{ix\xi} a(x,\xi) \hat{u}(\xi)\, d\xi.$$] Pseudo diff operators.

no longer = some polynomial.

- Approx. estimates · ~~can give~~ is one set of tools to invert ~~A~~ $E - R$.

- Parametrix ~ approximate inverse. $P \rightsquigarrow A$. "$= P^{-1}$".

  $Pu = f \Rightarrow u = Af$.
  properties of A give regularity.

- 

  Interior regularity problem.

  Singular problem, but bdry manifold.

Can change this ~~[illegible]~~ interior reg. problem to Pseudo-diff problem on bdry.